Math 311
Midterm Exam——October, 2016

Problem 1 (20 points): Determine whether the following statements are true or false?
(1): Let $\mathbb{A}$ be the set of all irrational numbers. Then $\mathbb{A}$ is an uncountable set.
(2): Let $\{A_j\}_{j\in\mathbb{N}}$ be a set of countable sets, where $\mathbb{N}$ is the set of nature numbers. Then $\cup_{j\in\mathbb{N}}A_j$ is also countable.
(3): Let $\{a_n\}$ be a Cauchy sequence. Then it is a convergent sequence.
(4): Let $\{a_n\}$ be a bounded sequence. Then it is a convergent sequence.
(5): Let $\{a_n\}$ be a sequence. If its two subsequences $\{a_{2k}\}_{k=1}^{\infty}$ and $\{a_{2k+1}\}_{k=0}^{\infty}$ are both convergent to the same number, then $\{a_n\}$ is convergent
(6): Let $S \subset \mathbb{R}$ be a bounded non-empty subset. Let $t = \sup S$, namely, the least upper bound of S. Then for any positive number ϵ, there is an element $s \in S$ such that $t - \epsilon \leq s \leq t$.
(7): Suppose $\lim_{x\to x_0} f(x) = 0$ and $g(x)$ is bounded. Then $\lim_{x\to x_0} f(x)g(x) = 0$.
(8): Every bounded subset of real numbers has only one greatest lower bound.
(9): An increasing sequence with an upper bound is convergent.
(10): The set of accumulation points of the open interval $(-1, \infty)$ consists of only one point $\{-1\}$.

Problem 2 (15 points): Prove by definition:

$$\lim_{n\to\infty} \frac{3n-2}{n} = 3.$$

Problem 3 (15 points) Prove by definition that

$$\lim_{x \to 1} (x^2 + 2x + 1) = 4.$$

Problem 4 (15 points): Let $a_1 = 4$ and $a_n = \sqrt{4 + a_{n-1}}$ for $n > 1$. Prove $\{a_n\}$ is a convergent sequence. Also, find its limit.

Problem 5 (10 points) Find the following limits:

$$(a): \lim_{n\to\infty} \frac{3n^3 - n + 100}{n^3 - n + 2}, \qquad (b): \lim_{x\to 0} \frac{\sqrt{x+4} - 2}{x}.$$

problem 6: (15 points). Suppose $f(x) : D \to \mathbb{R}$ be a function with $x_0 \in D$ as an accumulation point of D. Suppose that $\lim_{x \to x_0} f(x) = 2$. Prove that there is a positive number δ such that for any $x \in (D \setminus \{x_0\}) \cap (x_0 - \delta, x_0 + \delta)$, $f(x) > 1$.

problem 7: (15 points): State and prove the Bolzano-Weierstrass theorem.