This examination booklet contains 10 questions on 14 pages of paper including the front cover.

Do all of your work in this booklet, show all your computations and justify/explain your answers.

Do not remove any pages.

Your justification must be based on techniques already discussed in this course. If asked to evaluate an integral, remember to show all the steps that gives you its value.

Except for your personal note sheet, no other resources like class notes, books, calculator, etc are allowed. Remember that your note sheet must be handwritten, on both sides of a single sheet of paper.

Unless otherwise stated, give exact answers. For example, write $\pi$ and $\sqrt{2}$ instead of 3.14 and 1.41. However, when an expression simplifies to a well-known value, you must use that value. For example, you must write 1 instead of $e^0$, and you must write $\pi/3$ instead of $\sec^{-1}(2)$.

If you run out of space when answering a problem, you may use any of the last three pages of the exam, but you must: indicate in the space below the question that you are continuing your answer on the extra sheet, and indicate on the extra sheet which problem you are working on.

If asked to use a specific theorem to calculate some quantity, you will receive no points if that theorem is not used.

Do not discuss the exam with anyone until grades are posted on Canvas.

WRITE OUT AND SIGN PLEDGE

On my honor, I have neither received nor given any unauthorized assistance on this examination.
Problem 1. [12 points] Consider the curve given by the equation
\[ \mathbf{r}(t) = (t^3 + 1)\mathbf{i} + (t^2 - 8t)\mathbf{j} \]
Use this information to answer the following items (which are independent of each other).

a) Find the value(s) of \( t \) where the velocity vector of \( \mathbf{r}(t) \) is orthogonal (perpendicular) to the vector \( \overrightarrow{PQ} \), where \( P = (2, -1) \) and \( Q = (4, 5) \).

b) Compute the unit tangent vector \( \mathbf{T}(t) \).
Problem 2. [20 points] A closed rectangular plate $\mathcal{D}$ in the first quadrant is bounded by the lines $x = 0$, $x = 2$, $y = 0$ and $y = 1$. The temperature at the point $(x, y)$ on the plate is given by the function

$$T(x, y) = x^2 - 2x + y^3 - y$$

a) Find the critical point of $T(x, y)$ located in the interior of the rectangular plate, and classify it using the second derivative test.

b) Which important fact stated up front in section 14.7 of the book tells you that $T(x, y)$ takes on an absolute maximum and an absolute minimum on the plate $\mathcal{D}$? Recall that section 14.7 of the book was called “Optimization in Several Variables”.

c) Determine the hottest point(s) on the closed rectangular plate. **Hint:** notice that $T(x, y)$ can be written as $T(x, y) = x(x - 2) + y(y + 1)(y - 1)$.
Problem 3. [16 points] Consider the function

\[ V(x, y, z) = (x - 1)^2 + 2(y + 1)^2 + 2(z + 1)^2 \]

a) Compute the gradient of \( V(x, y, z) \).

b) Consider the level surface \( V(x, y, z) = 16 \). Find the points \( P_1 \) and \( P_2 \) on this level surface where the corresponding tangent plane to the surface has a normal vector pointing in the direction determined by the vector \( \langle 0, 1, 1 \rangle \). Choose \( P_1 \) to be the point with all its coordinates positive.

c) Consider the vector field \( \mathbf{F} = \nabla V \). Find the work done by \( \mathbf{F} \) along the line segment from \( P_1 \) to \( P_2 \).
Problem 4. [12 points] Consider a differentiable function \( f(u, v) \), which satisfies that

\[
\left. \frac{\partial f}{\partial u} \right|_{u=13, v=4} = \left. \frac{\partial f}{\partial v} \right|_{u=13, v=4} = 1
\]

Furthermore, assume that the variables \( u \) and \( v \) depend on \( x \) and \( y \) via the equations

\[
u = 2x + 3y, \quad v = -x + 2y
\]

and thus we can regard \( f \) as a function of \( x \) and \( y \).

Use the chain rule and the information above to find the directional derivative of \( f(x, y) \), with respect to \( x \) and \( y \), in the direction

\[
u = \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j
\]

at the point \((x, y) = (2, 3)\).
Problem 5. [16 points] Let $Q$ be the quadrilateral with vertices $(0, 0), (1, 4), (5, 0)$ and $(5, 4)$ in the $xy$ plane. Use a double integral to calculate the volume of the solid region above $Q$ and below the plane $z = 4x - y$. 
Problem 6. [18 points] Let $\mathcal{T}$ be a (triangular) pizza slice from Joe’s Pizza Place with vertices $(2,0,0)$, $(0,3,0)$, $(0,0,6)$. If cheese is distributed on the slice according to the density function $\delta(x,y,z) = z$ ounces per square inch, use a surface integral to calculate the amount of cheese on the pizza. That is, evaluate the surface integral

$$\iint_{\mathcal{T}} z \, d\sigma$$

When parametrizing $\mathcal{T}$, use $x$ and $y$ as parameters.
Problem 7. [16 points] Let $\mathcal{R}$ be the region in space inside the sphere $x^2 + y^2 + (z - 2)^2 = 4$ and outside the sphere $x^2 + y^2 + (z - 1)^2 = 1$. Using spherical coordinates, evaluate

$$\iiint_{\mathcal{R}} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$$
Problem 8. [18 points] Let $D$ be the planar region above the line $y = 1$ and inside the circle $x^2 + y^2 = 4$. Use Green’s Theorem to calculate the circulation of $F(x, y) = (e^x, xy^2)$ around the boundary of $D$, computed in counterclockwise sense.
Problem 9. [18 points] Let $S$ be the sake cup surface given by the parametrization

$$x = \sin h \cos \theta, \quad y = \sin h \sin \theta, \quad z = h^2$$

for $0 \leq \theta \leq 2\pi$, $0 \leq h \leq \frac{\pi}{2}$.

Use Stokes’ Theorem to calculate the outward flux of the curl of $F = \langle -ye^{-z}, xe^{-z}, \sqrt{x^2 + y^2} \rangle$ through $S$. 

![Diagram of a sake cup surface]
Problem 10. [24 points] Let $\mathcal{R}$ be the region in space below the paraboloid $z = 16 - x^2 - y^2$ and above the $xy$-plane. Call $\mathcal{S}$ the boundary of $\mathcal{R}$.

For the vector field $\mathbf{F}(x, y, z) = (x, y, 1)$, verify the result of the Divergence Theorem by calculating the outward flux of $\mathbf{F}$ through the boundary of $\mathcal{R}$ in two ways:

i) Using the definition of outward flux as the surface integral(s) $\iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, d\sigma$.

ii) By calculating instead the triple integral $\iiint_{\mathcal{R}} \nabla \cdot \mathbf{F} \, dV$ which appears in the Divergence theorem.
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