(1) (a) (5 points) State the definition of the dimension of a vector space.

Ans: The dimension of a vector space is the number of elements in any basis. The dimension of a vector space is not the number of elements in “the” basis. There is not just one basis. The dimension of a vector space is not the number of linearly independent vectors. A vector by itself is not linearly independent or dependent; the adjective “linearly independent” applies to a set of vectors.

(b) (10 points) Find a basis for the vector space of polynomials $f(x)$ of degree at most 2 satisfying $f'(1) = 0$. What is the dimension of this vector space? Show your work.

Ans: Let $f(x) = a + bx + cx^2$ be a polynomial of degree at most 2, where $a, b, c$ are arbitrary real numbers. Then $f'(x) = b + 2cx$ so $f'(1) = b + 2c$. So $f'(1) = 0$ means that $b + 2c = 0$. This is a linear system with one equation and three unknowns, with matrix form $[0 \ 1 \ 2 | 0]$. The pivot is in the second column, so $a, c$ are free and $b + 2c = 0$ so $b = -2c$. So the general form of the solution is $f(x) = a - 2cx + cx^2 = a + c(x^2 - 2x)$. This is a linear combination of the functions 1 and $x^2 - 2x$, with scalars $a$ and $c$. These are clearly independent, since they are not scalar multiples of each other, so a basis for the space of solutions is 1, $x^2 - 2x$.

(2) (a) (5 points) State the definition of a linearly independent subset (not necessarily finite) of a vector space.

Ans: A subset $S$ of a vector space is linearly independent iff for all scalars $c_1, \ldots, c_n$ and vectors $v_1, \ldots, v_n$ in $S$, if $c_1 v_1 + \ldots + c_n v_n = 0$ then $c_1 = c_2 = \ldots = c_n = 0$.

(b) (15 points) True or false: the functions $\sin(x), \sin(2x)$ are linearly independent as functions of a real variable $x$. Prove your answer.

Ans: Suppose that $c_1 \sin(x) + c_2 \sin(2x) = 0$ for some scalars $c_1, c_2$.

Plugging in $x = 0$ just gives $c_1 0 + c_2 0 = 0$, so $0 = 0$, which doesn’t help.

Plugging in $x = \pi/2$ gives $c_1 1 + c_2 0 = 0$ so $c_1 = 0$.

Plugging in $x = \pi$ doesn’t help, since it gives $0 + 0 = 0$ again.

Plugging in $x = \pi/4$ gives $c_1 / \sqrt{2} + c_2 = 0$. Since $c_1 = 0$, we have $c_2 = 0$ as well.

So $c_1 \sin(x) + c_2 \sin(2x) = 0$ for some scalars $c_1, c_2$ implies $c_1 = c_2 = 0$.

So the functions are linearly independent.

Another cool solution was:

Ans: $\sin(2x) = 2 \sin(x) \cos(x)$. To say that $\sin(2x)$ is a scalar multiple of $\sin(x)$ would be to say that $2 \cos(x)$ is a scalar (constant), which it is not. So $\{\sin(x), \sin(2x)\}$ are linearly independent.

(3) (a) (15 points) Show that the set of polynomials $f(x)$ satisfying $f(x) = f(-x)$ for all $x$ is a subspace of the vector space of functions of a real variable.
Ans: For a set to be a subspace means that any two elements in the set add to an element of the set, and the scalar multiple of anything in the set is in the set, and 0 is in the set.

(SS1) Let’s take two elements of the set, which is the set of even polynomials. We want to show that the sum of even polynomials is also even. Let \( f, g \) be even polynomials. Then \( f + g \) is even iff \( (f + g)(-x) = (f + g)(x) \) for all \( x \). The left hand side is \( f(-x) + g(-x) = f(x) + g(x) = (f + g)(x) \). So \( f + g \) is also even.

(SS2) If \( f \) is even and \( c \) is a scalar then \( cf \) is even iff \( (cf)(x) = (cf)(-x) \). But \((cf)(-x) = c(f(-x)) = c(f(x)) = (cf)(x)\).

(SS3) Finally let \( 0(x) = 0 \) be the zero function. Then \( 0(-x) = 0 = 0(x) \) so \( 0 \) is even.

(b) (15 points) Find the dimension of the subspace in part (a), and, if possible, a basis. Justify your answer.

Any polynomial is of the form \( f(x) = a_0 + a_1x + \cdots + a_nx^n \) for some reals \( a_1, \ldots, a_n \). Then \( f(-x) = a_0 - a_1x + a_2x^2 - a_3x^3 \ldots \) So \( f(x) = f(-x) \) iff (matching coefficients) \( a_0 = a_0, a_1 = -a_1, a_2 = a_2, \ldots \), which holds iff \( a_1 = 0, a_3 = 0, a_5 = 0, \ldots \).

So the general form of an even polynomial is \( a_0 + a_2x^2 + a_4x^4 + \ldots + a_{2n}x^{2n} \) where \( n \) is a non-negative integer and \( a_0, a_2, a_4, \ldots, a_{2n} \) are real numbers.

So the polynomials \( B = \{1, x^2, x^4, \ldots \} \) span \( V \).

We proved in class that these are linearly independent so they are a basis. Since \( B \) has infinite size, the dimension of \( V \), which is the size of \( B \), is infinite.

(4) Let \( V = W = \text{span} P_3 \) and \( B = \{1, x, x^2, x^3\} \) the standard basis. Let \( T \) be the linear transformation \( T(f) = f(2x) - 2f(x) \).

(a) (10 points) Find the matrix of \( T \) with respect to \( B \).

Ans: To find the matrix, we apply \( T \) to each basis element and take coordinates. If \( f(x) = 1 \) then \( f(2x) - 2f(x) = 1 - 2 = -1 \) which has coordinates \(-1, 0, 0, 0\). If \( f(x) = x \) then \( f(2x) - 2f(x) = 2x - 2x = 0 \) which has all coordinates \(0\). If \( f(x) = x^2 \) then \( f(2x) - 2f(x) = (2x)^2 - 2x^2 = 2x^2 \) which has coordinates \(0, 0, 2, 0\). If \( f(x) = x^3 \) then \( f(2x) - 2f(x) = 6x^3 \) which has coordinates \(0, 0, 0, 6\). So the matrix of \( T \) is diagonal with diagonal entries \(-1, 0, 2, 6\).

(b) (10 points) Find a basis for the nullspace of \( T \).

Ans: \( N(T) = \{f(Tf) = 0\} = \{f(Tf)(x) = 0, \text{ all } x\} \).

So \( N(T) \) is the set of functions \( f(x) \) such that \( f(2x) - 2f(x) = 0 \), that is \( f(2x) = 2f(x) \), for all \( x \).

Let \( f(x) = a + bx + cx^2 + dx^3 \). Then \( (Tf)(x) = -a + 2cx^2 + 6dx^3 \). So \( Tf = 0 \) iff \( a = c = d = 0 \). The variable \( b \) is free so \( f(x) = bx \). So \( N(T) = \{bx, b \text{ free}\} = \text{span}\{x\} \) (that is, the span of function given by \( f_1(x) = x \)).

So \( \{x\} \) is a basis.
(c) (10 points) Find a basis for the range of $T$.

Ans. We already computed that \( \{Tf\} = \{-a + 2cx^2 + 6dx^3, \ a,c,d \ free\} = \text{span}\{1, x^2, x^3\} \). So \( B = \{1, x^2, x^3\} \) is a basis.

(d) (5 points) Is $T$ an invertible linear transformation? Why or why not?

$T$ is not one-to-one, since it has a null-space, so it is not invertible. Also the matrix is not invertible, so $T$ is not invertible. Also $T$ is not onto, since it is not maximal rank, so $T$ is not invertible.