Preparation for Math 152

This packet was originally designed for students transitioning
from:
Math 135: Calculus I for the Life and Social Sciences
to:
Math 152: Calculus II for the Mathematical and Physical Sciences

It will also be good practice for any other student entering Math 152!

This packet grew out of my experience teaching and coordinating Math 158, a 1-credit practicum designed as a helper course to be taken alongside Math 152. I hope someone finds this packet helpful. Special thanks to Laurent Vera, Malcolm Bowles, Emily Sergel, Matt Sequin, and Peter Ullman for many helpful comments, corrections, and suggestions. Feel free to email me with comments, questions, typos, suggestions, etc:
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Contents

Section 1 (Trig Review) is an important topic that many students need to review before any calculus class.

Sections 2, 3, 4, 5 are very short topics covered in Math 151, but not in Math 135.

Section 6 (Substitution) is covered in Math 135, but not as deeply as in Math 151. The extra practice provided by this packet is vital for success in Math 152. This topic will come up over and over again in many forms.

Section 7 (Area Between Curves) is covered in Math 151, but not in Math 135 at all. It is the foundation for many methods in Math 152.

Section 8 covers the six Inverse Trig Functions. It lays their precalculus foundations from the ground up, but also shows how to find their derivatives. You can treat their derivatives as formulas to memorize when you first start Math 152, but eventually, you will really need to understand these functions.

Section 9 is a review of limit calculations, along with some new methods that are not covered in Math 135 or Math 151. This is not needed for Math 152 until later in the semester. However, it is best to practice these methods before they are needed.

Section 10 (Sigma Notation) is also not needed until later in Math 152. It is also not covered at all in Math 135. Again, it is best to practice before it is needed.

Suggested Use

- Before the semester starts:
  - Make sure Section 1 is 100% rock solid
  - Read, understand, take notes, and do the problems from on Sections 2, 3, 4, 5. These are short
  - Do as many problems from Sections 6 and 7 as you can. Mix it up. Do some of each. These are the parts that are most directly relevant to the first part of Math 152.
  - Start Section 8. At least make note of the derivative formulas for now so that you can apply them right away.
- Within the first 2 weeks of the semester:
  - Finish the problems in Sections 6 and 7.
  - Finish Section 8.
- Within the first 4 weeks of the semester:
  - Sections 9 and 10

1 Trig Review

You really gotta know your trig. How much? Good question...

Summary of trig that you need for Math 152

Definitions:

$\sin x$ and $\cos x$ are defined using the unit circle. After that, the other 4 trig functions are defined using $\sin x$ and $\cos x$ as follows:

$$\tan x = \frac{\sin x}{\cos x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \csc x = \frac{1}{\sin x}$$

The sine function is odd and the cosine function is even:

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x$$

Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Double Angle Identities:

$$\sin(2x) = 2\sin x\cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

Manipulate the previous ones to obtain these:

$$\sin^2(x) = \frac{1-\cos 2x}{2} \qquad \cos^2(x) = \frac{1+\cos 2x}{2}$$

Trig and Calc

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x\tan x$$

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0 \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x\cot x$$

Unit Circle

Exercise 1. Make sure you can fill out this chart in 10 minutes or less:

x	0	π	$-\pi$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{7\pi}{6}$	$\frac{-4\pi}{3}$	$\frac{7\pi}{4}$
$\sin x$										
$\cos x$										
$\tan x$										
$\sec x$										
$\csc x$										
$\cot x$										

Trig Graphs

Exercise 2. Make sure you can sketch the graphs of $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \sec x$ fairly accurately.

Need a trig refresher? Everything above is explained in this video series:

`https://www.youtube.com/playlist?list=PLNx0uUvEWthPT3Knhm85SabXxB4KJNFYZ`

2 Derivatives of unnatural logarithms

You should recall the formulas: $\boxed{\frac{d}{dx}\left(e^x\right) = e^x \qquad \text{and} \qquad \frac{d}{dx}\left(\ln x\right) = \frac{1}{x}}$

But what about, for example, $\frac{d}{dx}\left(7^x\right)$ or $\frac{d}{dx}\left(\log_6 x\right)$? It turns out that these differ from the above formulas by a predictable constant:

$$\frac{d}{dx}\left(7^x\right) = \frac{d}{dx}\left(\left(e^{\ln 7}\right)^x\right) = \frac{d}{dx}\left(e^{(\ln 7)x}\right) = e^{(\ln 7)x}\ln 7 = \left(e^{\ln 7}\right)^x \ln 7 = 7^x \ln 7$$

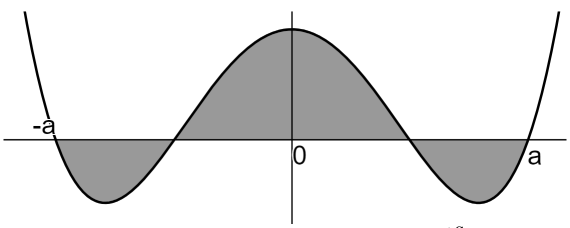
$$\frac{d}{dx}\left(\log_6 x\right) = \frac{d}{dx}\left(\frac{\ln x}{\ln 6}\right) = \frac{d}{dx}\left(\frac{1}{\ln 6}\ln x\right) = \frac{1}{x \ln 6}$$

New formulas: $\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a \qquad \text{and} \qquad \frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}}$

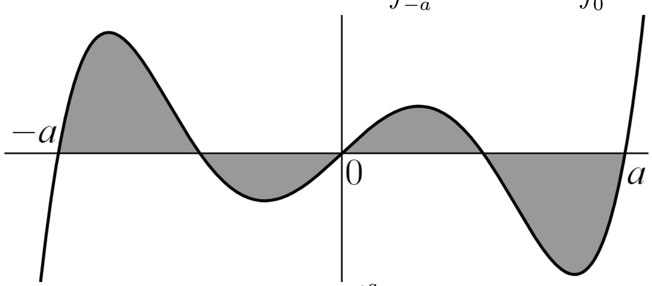
3 Definition of the Integral

Oh, just watch this video: `https://www.youtube.com/watch?v=MwVBzE7Z5gw`

4 Even and Odd functions



If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$



If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$

Exercise 3. Find $\int_{-23}^{23} \sin\left(x^5\right)\,dx$

5 Average Value

Consider the video in Section 3. It makes the point that integration is like adding up all the values of a function $f(x)$ on on interval $[a, b]$. How can we use this to understand the average value of a function on a given interval? One way is to assert that we should get the same result if we had instead added the average value of $f(x)$ the whole time. In other words

$$\begin{aligned}\int_a^b f(x)\,dx &= (\text{width of the interval})(\text{average value of } f(x)) \\ &= (b-a)(\text{average value of } f(x))\end{aligned}$$

That is:

$$\text{Average value of } f(x) \text{ on the interval } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Exercise 4. What is the average value of $f(x) = x^2$ on the interval $1 \le x \le 5$?

6 Substitution

Most Calculus II courses have four main parts:

- Methods of Integration
- Applications of the Integral
- Sequences and Series
- Parametric and Polar Coordinates

In this section we explore one method of integration, substitution. The idea behind substitution will come up several times in your Calculus II course. First let's make sure we agree on some basic rules:

— Review of antiderivative rules

These are based on the analogous derivative rules:

$$\int 0\ dx = C$$

$$\int 54\ dx = 54x + C$$

$$\int x\ dx = \frac{x^2}{2} + C$$

$$\int x^2\ dx = \frac{x^3}{3} + C$$

$$\int x^3\ dx = \frac{x^4}{4} + C$$

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$

(except when $n = -1$)

$$\int \cos x\ dx = \sin x + C$$

$$\int \sin x\ dx = -\cos x + C$$

$$\int \sec^2 x\ dx = \tan x + C$$

$$\int \sec x \tan x\ dx = \sec x + C$$

$$\int -\csc^2 x\ dx = \cot x + C$$

$$\int -\csc x \cot x\ dx = \csc x + C$$

$$\int \frac{1}{x}\ dx = \ln|x| + C$$

$$\int e^x\ dx = e^x + C$$

$$\int a^x\ dx = \left(\frac{1}{\ln a}\right) a^x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C$$

(These last three formulas are covered in Section 8.)

Sums: $$\int [f(x) + g(x)]\ dx = \int f(x)\ dx + \int g(x)\ dx$$

Differences: $$\int [f(x) - g(x)]\ dx = \int f(x)\ dx - \int g(x)\ dx$$

Constant Multiples: $$\int [bf(x)]\ dx = b\int f(x)\ dx \text{ (if } b \text{ is a constant)}$$

Now we can, for example, find the antiderivative of any polynomial:

$$\int \left(34x^6 - 4.3x^5 + \frac{\pi}{2}x^2 - x - 34\right) dx = \frac{34x^7}{7} - \frac{4.3x^6}{6} + \frac{\pi x^3}{6} - \frac{x^2}{2} - 34x + C$$

You can check your answer for this kind of problem: differentiate the right side and see if you get the left side!

What about $\int f(x)g(x)\ dx$? Turn the page to find out.

— There is no product rule for integrals

Strictly speaking, 'Methods of Integration' should be called 'Methods of Antidifferentiation.' But since we use these antiderivatives along with the Fundamental Theorem to calculate integrals, the traditional name also makes sense. All of the methods that are discussed in Calculus II are partial solutions to the same problem:

There is no Product Rule for Integrals!

In other words, the methods we have so far are not enough to calculate something like $\int 2x\cos(x)\,dx$. You may be tempted to guess that the antiderivative is $x^2\sin(x)+C$, but that would be wrong! The Product Rule tells us that the derivative of $x^2\sin(x)$ is $x^2\cos(x)+2x\sin(x)$. Instead, the correct answer is $\int 2x\cos(x)\,dx = 2x\sin(x)+2\cos(x)+C$.[1] (Check this answer by differentiating it!)

— The Method of Substitution

So there is no product rule for integrals, but there are some methods that work in certain situations. The method of this section is called substitution, and it is a way of "undoing" the Chain Rule. So let's take a look at this example of the Chain Rule in action:

$$\left[\sin(x^2)\right]' = \cos(x^2)\cdot(x^2)' = 2x\cos(x^2)$$

This of course means that

$$\int 2x\cos(x^2)\,dx = \sin(x^2)+C$$

But what if you didn't know this, and were faced with $\int 2x\cos(x^2)\,dx$? How would you know that this was a case of the Chain Rule? If we look at exactly how the Chain Rule was used, we see that we had sin of the function x^2, and so the derivative was cos of x^2 times the derivative of x^2. In order to "undo" this process we need to identify the "inside" function, which in our case is x^2. So back to solving $\int 2x\cos(x^2)\,dx$: we make an educated guess and mark the function x^2 by setting it equal to a new variable, let's say u: $u=x^2$ But since the derivative of this "inner" function is also used in the Chain Rule, we also have to mark out its derivative: $du = 2x\,dx$. Putting these together we get:

$$\int \underline{2x}\,\underline{\cos(x^2)}\,\underline{dx} = \int \underline{\cos(u)}\,\underline{du}$$

but then we know how to solve this antiderivative:

$$\int 2x\cos(x^2)\,dx = \int \cos(u)\,du = \sin(u)+C$$

and then finally, we substitute our original variable x back in:

$$\int 2x\cos(x^2)\,dx = \int \cos(u)\,du = \sin(u)+C = \sin(x^2)+C$$

which is the answer we were looking for.

Example. Find $\int (4x+2)\sqrt{x^2+x}\,dx$.

If we let $u = x^2+x$, then $du = (2x+1)\,dx$, and so $2du = (4x+2)\,dx$.

$$\begin{aligned}
\int (4x+2)\sqrt{x^2+x}\,dx &= \int \sqrt{u}(2du) = 2\int \sqrt{u}\,du \\
&= 2\int u^{\frac{1}{2}}\,du \\
&= 2\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C \\
&= \frac{4}{3}u^{\frac{3}{2}}+C \\
&= \frac{4}{3}(x^2+x)^{\frac{3}{2}}+C
\end{aligned}$$

[1] In Math 152 you will solve this using the method of Integration by Parts.

— Multiplying the variable by a constant:

Exercise 5. $\displaystyle\int e^{3x}\,dx$

Exercise 6. $\displaystyle\int \cos 4x\,dx$

Exercise 7. Using the previous two exercises as inspiration, find a general formula for $\displaystyle\int \sin ax\,dx$, where a is a constant.

From now on if the variable is multiplied by a constant, you are expected to be able to do the substitution in your head.

— One substitution. Just let u be the 'inside function':

Exercise 8. $\displaystyle\int \frac{4x^3}{(x^4+1)^{10}}\,dx$ **Exercise 9.** $\displaystyle\int x\sin(2x^2)\,dx$ **Exercise 10.** $\displaystyle\int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx$

— Still just one substitution but now the choice of u is less obvious:

Exercise 11. $\displaystyle\int \frac{1}{x\ln x}\,dx$ **Exercise 12.** $\displaystyle\int \tan x\,dx$ **Exercise 13.** $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

— Substitution with the Definite Integral

Let's find $\displaystyle\int_3^4 \sin(5x)\,dx$. Since the antiderivative is $-\frac{1}{5}\cos(5x)+C$, The Fundamental Theorem of Calculus tells us that this definite integral is equal to $\left[-\frac{1}{5}\cos(5\cdot 4)\right]-\left[-\frac{1}{5}\cos(5\cdot 3)\right]$.

However, there is a shortcut we can take while doing the substitution. Instead of rewriting everything in terms of x at the end of the problem, we can leave it in terms of u, but make sure to also write the values for the endpoints also in terms of u.

As before, if we let $u=5x$, then $du=5dx$, and so $dx=\frac{1}{5}du$. This time, note that the endpoint $x=3$ becomes $u=15$, since $u=5x$. Similarly, the endpoint $x=4$ becomes $u=20$.

$$\begin{aligned}\int_{x=3}^{x=4}\sin(5x)dx = \int_{u=15}^{u=20}\sin u\left(\frac{1}{5}du\right) &= \frac{1}{5}\int_{15}^{20}\sin u\,du\\ &= -\frac{1}{5}\cos u\Big|_{15}^{20}\\ &= \left[-\frac{1}{5}\cos(20)\right]-\left[-\frac{1}{5}\cos(15)\right]\end{aligned}$$

And this is the answer we got above! This example uses the same method:

Example. Find $\displaystyle\int_5^6 (5x+3)^{700}dx$.

If we let $u=5x+3$, then $du=5dx$, and so $dx=\frac{1}{5}du$.

$$\begin{aligned}\int_{x=5}^{x=6}(5x+3)^{700}dx = \int_{u=28}^{u=33}u^{700}\left(\frac{1}{5}du\right) &= \frac{1}{5}\int_{28}^{33}u^{700}du\\ &= \frac{1}{5}\left[\frac{u^{701}}{701}\right]_{28}^{33}\\ &= \frac{1}{5}\left[\frac{33^{701}}{701}-\frac{28^{701}}{701}\right]\end{aligned}$$

— You must substitute the bounds as shown in the example above:

Exercise 14. $\displaystyle\int_0^4 e^{3x}\,dx$

Exercise 15. $\displaystyle\int_0^1 x\sqrt{1-x^2}\,dx$

Exercise 16. $\displaystyle\int_{-1}^1 x\sqrt{1-x^2}\,dx$

Exercise 17. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$

Exercise 18. $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x dx}{1+e^{2x}}$

Exercise 19. $\displaystyle\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^{\pi}}$

— Choose u, then solve for x. Or maybe x^2? Or x^4?

Exercise 20. $\displaystyle\int (x+1)\sqrt[5]{x}\,dx$

Exercise 21. $\displaystyle\int x\sqrt[5]{x-1}\,dx$

Exercise 22. $\displaystyle\int \frac{x^3}{\sqrt[3]{1-x^2}}\,dx$

Exercise 23. $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$

— Find $\int 2\sin x\cos x\,dx$

Exercise 24. Find $\displaystyle\int 2\sin x\cos x\,dx$. First try $u=\sin x$

Exercise 25. Find $\displaystyle\int 2\sin x\cos x\,dx$. Next try $u=\cos x$

Exercise 26. Find $\displaystyle\int 2\sin x\cos x\,dx$. Now use the identity $2\sin x\cos x=\sin 2x$ first.

Exercise 27. OMG, we got three different answers!!! How is this possible?!?! Try graphing the three solutions to see how they differ...

— All of these use the rule for the inverse sine function:

Exercise 28. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

Exercise 29. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

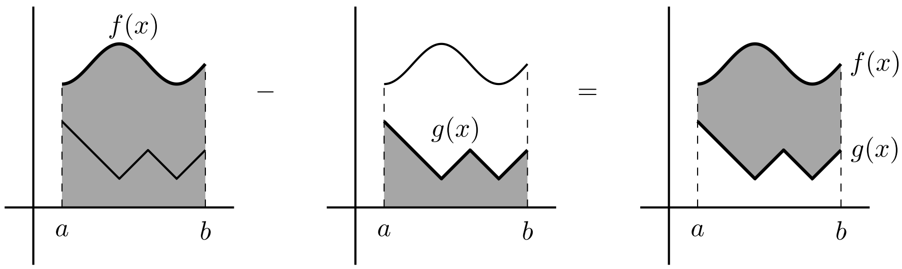
Exercise 30. $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^4}}$

Exercise 31. $\displaystyle\int \frac{dx}{\sqrt{8x-x^2}}$. Hint: complete the square first.

7 Area Between Curves

In this section we discuss a simple application of the integral, namely, finding the area between two graphs. Suppose $f(x)$ is greater than $g(x)$ between $x = a$ and $x = b$. Then the area *between* these curves on this interval is

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$



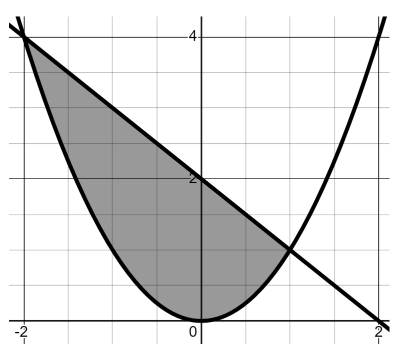
Example 1. Find the area between $f(x) = x^2$ and $g(x) = -x + 2$.
In this example, we are not given the endpoints, so we have to figure out where these graphs intersect, and also which one is greater than the other between the points of intersection. First, we find the points of intersection:

$$\begin{aligned} f(x) &= g(x) \\ x^2 &= -x + 2 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \end{aligned}$$

So the points of intersection are $x = -2$ and $x = 1$. In order to find out which function is greater between these points, we just take any test point between these values.[2] Let's use the test point $x = 0$. $f(0) = 0$ and $g(0) = 2$, so $g(x)$ is greater than $f(x)$ on this interval. So now we can find the area:

$$\begin{aligned} \int_{-2}^{1} [g(x) - f(x)]dx &= \int_{-2}^{1} [(-x+2) - (x^2)]dx \\ &= \int_{-2}^{1} [-x^2 - x + 2]dx \\ &= -\frac{x^3}{3} - \frac{x^2}{2} + 2x\Big|_{-2}^{1} \\ &= \left(-\frac{1^3}{3} - \frac{1^2}{2} + 2\cdot 1\right) - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2)\right) \\ &= \frac{9}{2} \end{aligned}$$

Just to illustrate what we did, let's look at the graph.

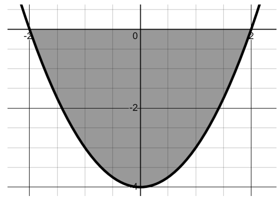


However, we did solve this problem without knowing what the graph looked like!

[2]Remember how you used test points to make sign charts of the first and second derivative?

A note on negative area

Consider the graph of the function $f(x) = x^2 - 4$. It has x-intercepts at $x = -2$ and $x = 2$:



First, let's look at the integral between these intercepts:

$$\int_{-2}^{2} (x^2 - 4)\, dx = \left(\frac{x^3}{3} - 4x\right)\Big|_{-2}^{2} = \left[\frac{2^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right] = -\frac{32}{3}$$

The fact that we get a negative number here means that the integral does not exactly model the *area* between the curve and the x-axis. And this makes sense, given how we defined the integral. After all, the rectangles we would use to define this integral would have 'negative heights', since we use the function value to determine heights and our function $f(x) = x^2 - 4$ is in fact negative between $x = -2$ and $x = 2$.

How do we make sense of this using the ideas of this section? The area we are talking about is the area between $f(x) = x^2 - 4$ and the x-axis, which has the equation $g(x) = 0$. We might not be used to thinking of the x-axis as a function with an equation, but it is! On this interval, the curve $g(x)$ is greater than $f(x)$, so our 'area between curves' method tells us that the area is

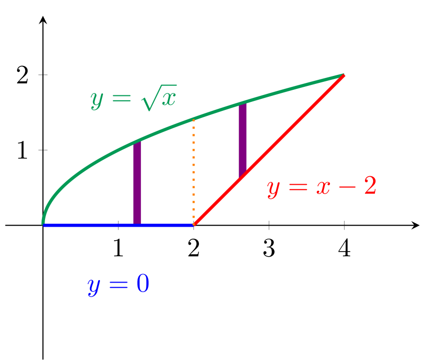
$$\int_{-2}^{2} \left[(0) - (x^2 - 4)\right]\, dx = \int_{-2}^{2} (4 - x^2)\, dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \left[4(2) - \frac{2^3}{3}\right] - \left[4(-2) - \frac{(-2)^3}{3}\right] = \frac{32}{3}$$

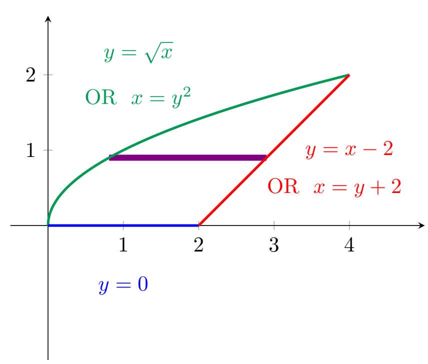
And, of course, this is the area between our function and the x-axis.

We conclude that, as long as we are measuring the area *between* curves, even if one of the curves happens to be the x-axis, then the integral *does* properly model the concept of area.

Integration with respect to y

Example 2. Find the area between $y = \sqrt{x}$, $y = 0$, and $x - 2$. Do it two ways, once with respect to x, and once with respect to y.



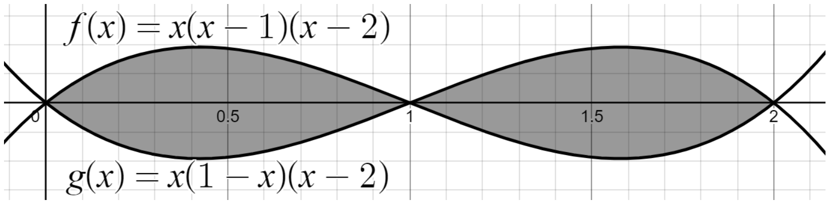


$$\begin{aligned}
\text{Area} &= \int_0^2 \sqrt{x}\, dx + \int_2^4 \left(\sqrt{x} - (x-2)\right)\, dx \\
&= \int_0^2 \sqrt{x}\, dx + \int_2^4 \sqrt{x}\, dx - \int_2^4 (x-2)\, dx \\
&= \int_0^4 \sqrt{x}\, dx - \int_2^4 (x-2)\, dx \\
&= \frac{16}{3} - 2 = \frac{10}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Area} &= \int_0^2 \left((y+2) - y^2\right)\, dy \\
&= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_0^2 \\
&= \left(2 + 4 - \frac{8}{3}\right) - (0 + 0 - 0) \\
&= \frac{10}{3}
\end{aligned}$$

Setting up integrals like this forms the foundation of all of the problems in Chapter 6. Practice!

Exercise 32. What integral or sum of integrals represents the area shown? **Do not solve.**



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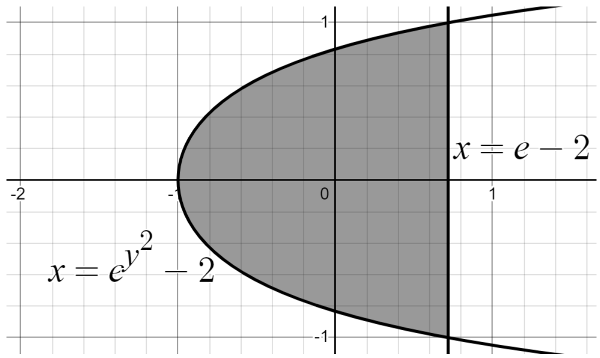
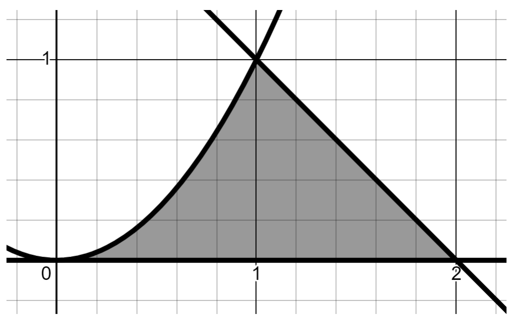


Exercise 34. Find the area between $y = (1 - \cos x)\sin x$ and the x-axis on the interval $0 \le x \le \pi$.

Exercise 35. A region between $y = x^2$, $x + y = 2$, and $y = 0$ is shown.



(a) Find the area by integrating over x.

(b) Find the area by integrating over y.

(c) Did you get the same area both times?

Find the areas of the regions enclosed by the curves:

Exercise 36. $y = x^2$, $y = x^3$

Exercise 37. $y = 5e^x$, $y = 5xe^{x^2}$, $x = 0$. Hint: the first two graphs only intersect at one point.

Exercise 38. $y = 2\sin x$, $y = \sin 2x$, $0 \le x \le \pi$

Exercise 39. $y = \arctan x$, $y = \frac{\pi}{4}$, $x = 0$

Exercise 40. $y = \ln x$, $y = \ln 2x$, $1 \le x \le 5$

Exercise 41. $y = x^2 - 2$, $y = |x|$

Exercise 42. $y = x^2$, $y = x^4 - 4x^2 + 4$ (**Set up the integral only**)

Exercise 43. Find b so that $y = b$ divides the region bounded by $y = x^2$ and $y = 4$ into two regions with equal area.

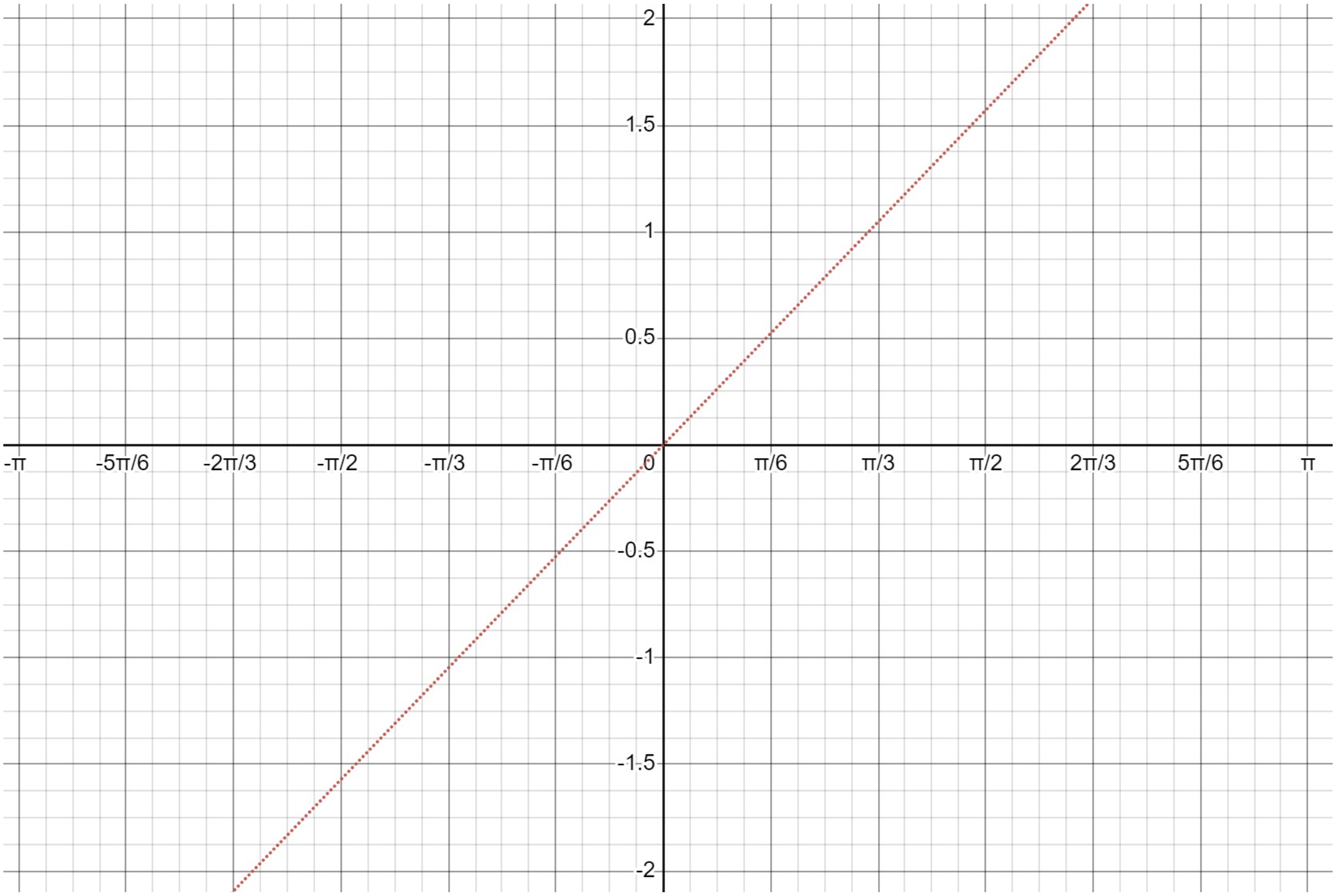
Exercise 44. Consider the triangle with vertices $(1, 4)$, $(2, 1)$, $(4, 2)$.

(a) Find the area by integrating over x.

(b) Find the area by integrating over y.

(c) Did you get the same area both times?

8 Inverse Trig Functions



Exercise 45. Constructing arcsin.

(a) On the graph above, *carefully* draw the graph of $y = \sin x$ on the domain $(-\pi, \pi)$

(b) Notice that the graph you drew does not pass the horizontal line test. So what domain should we restrict it to in order to construct an inverse function? Highlight the part of the graph on that domain.

(c) Reflect the highlighted portion of the graph across the line $y = x$, which is provided for you. Label the endpoints. Congratulations, you just drew the graph of $y = \sin^{-1} x$, also called $y = \arcsin x$

Exercise 46. Find the exact value of each of the following:

(a) $\sin^{-1}\left(\frac{1}{2}\right)$ (b) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ (c) $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$ (d) $\sin^{-1}(-1)$

(e) $\cos^{-1}\left(\frac{1}{2}\right)$ (f) $\arccos\left(\frac{\sqrt{3}}{2}\right)$ (g) $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ (h) $\cos^{-1}(-1)$

Exercise 47. Find the exact value of each of the following:

(a) $\sin\left(\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)\right)$ (b) $\sec\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$ (c) $\cot\left(\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ (d) $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right)$

(e) $\sin\left(\cos^{-1}\left(\frac{3}{7}\right)\right)$ (f) $\cot\left(\arcsin\left(3x\right)\right)$

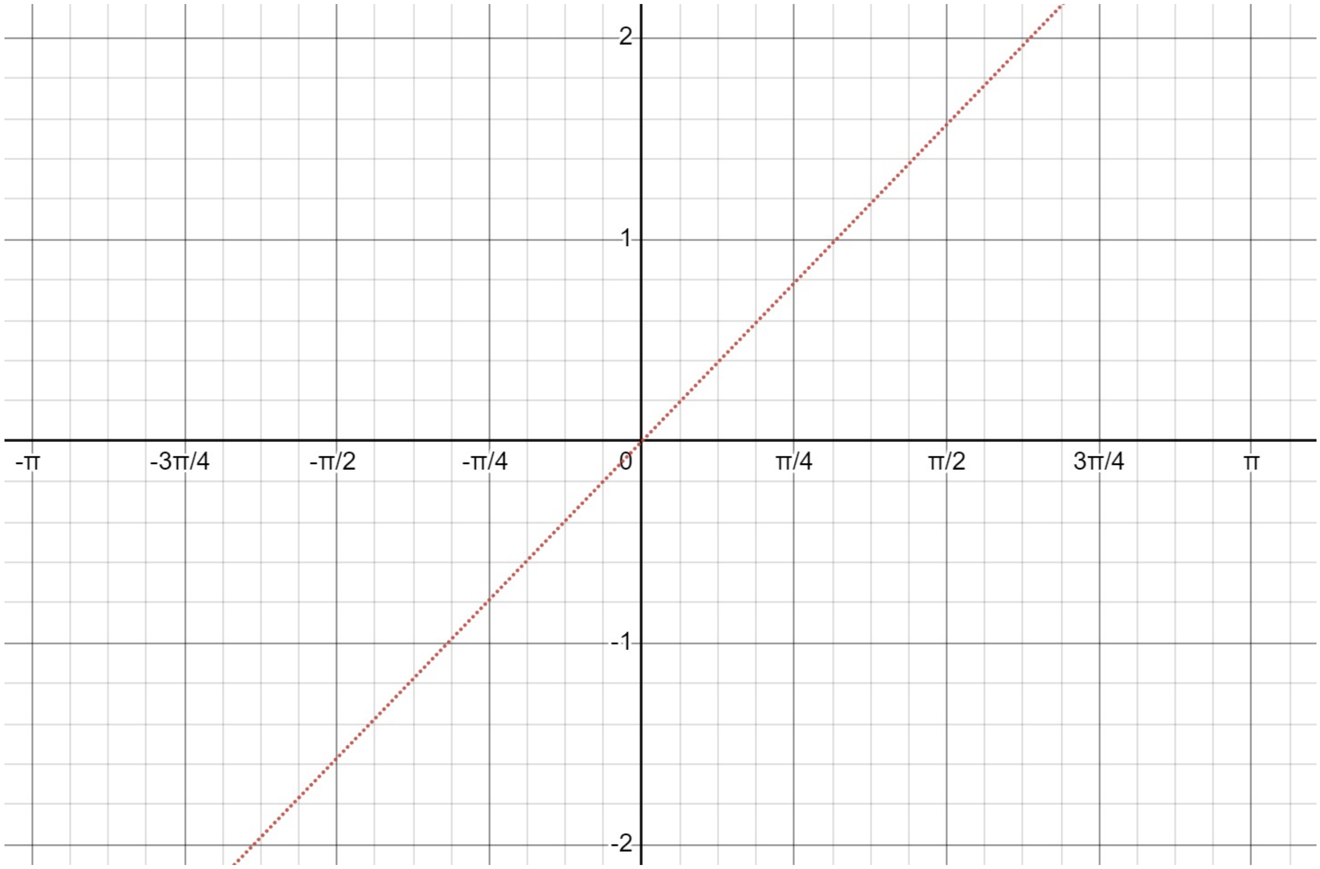
Exercise 48. Derivative of arcsin. First rewrite $\mathbf{y} = \sin^{-1}\mathbf{x}$ as $\sin\mathbf{y} = \mathbf{x}$. Then use implicit differentiation.

(a) Draw a right triangle with 1 as the hypotenuse and one of the legs as x.

Use this to explain why $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

(b) Rewrite this formula as $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$.

(c) Use this formula, together with the derivative of $y = \sin^{-1} x$, to find the derivative of $y = \cos^{-1} x$



Exercise 49. Constructing arctan.

(a) On the graph above, *carefully* draw the graph of $y = \tan x$ on the domain $(-\pi, \pi)$

(b) Notice that the graph you drew does not pass the horizontal line test. So what domain should we restrict it to in order to construct an inverse function? Highlight the part of the graph on that domain.

(c) Reflect the highlighted portion of the graph across the line $y = x$, which is provided for you. Label the asymptotes. Congratulations, you just drew the graph of $y = \tan^{-1} x$, also called $y = \arctan x$

Exercise 50.

(a) What is the domain of $y = \tan^{-1} x$?

(b) What is the range of $y = \tan^{-1} x$?

(c) $\lim_{x \to -\infty} \tan^{-1} x$

(d) $\lim_{x \to \infty} \tan^{-1} x$

Exercise 51. Find the exact value of each of the following:

(a) $\tan^{-1} 0$ (b) $\tan^{-1} 1$ (c) $\tan^{-1} -1$ (d) $\tan^{-1} \sqrt{3}$

Exercise 52. Derivative of arctan. Use the same method as Exercise 48 to find the derivative of $y = \tan^{-1} x$.

Exercise 53. Derivative of arcsec. Use the same method as Exercise 48 to find the derivative of $y = \sec^{-1} x$.

Exercise 54. Find derivatives of the following functions:

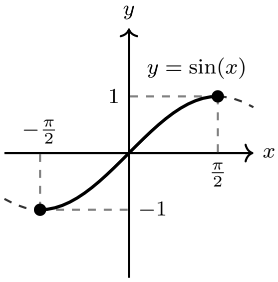
(a) $y = x \tan^{-1}(x) - \dfrac{1}{2} \ln(4x^2 + 4)$ (b) $y = x \sin^{-1} x + \sqrt{1 - x^2}$

(c) $y = \cot^{-1} \left(\dfrac{1}{x} \right) - \tan^{-1} x$ (d) $y = \tan^{-1} \sqrt{x^2 - 1} + \csc^{-1} x$ if x is positive

Inverse Trig Summary

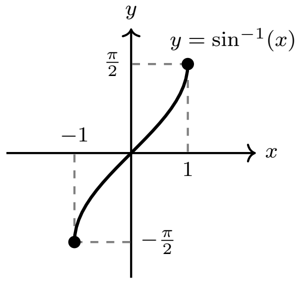
You are responsible for knowing all of the following information.

Restricted sine



- Domain: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- Range: $[-1, 1]$

Arcsine

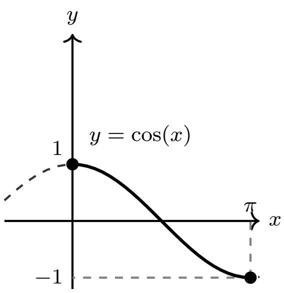


- Domain: $[-1, 1]$
- Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Derivatives:

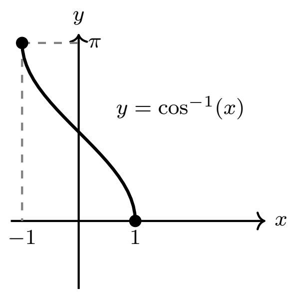
- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

Restricted cosine



- Domain: $[0, \pi]$
- Range: $[-1, 1]$

Arccosine

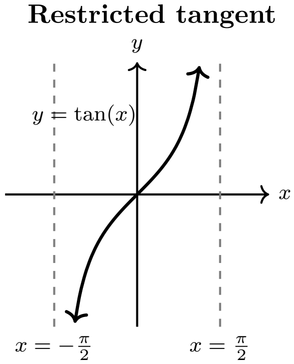


- Domain: $[-1, 1]$
- Range: $[0, \pi]$

Derivatives:

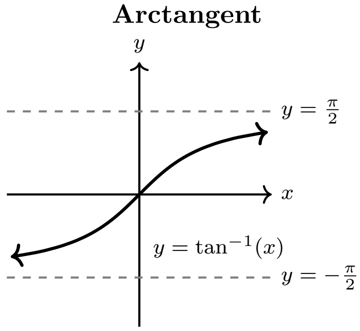
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- $\dfrac{d}{dx}(\sin^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$

Restricted tangent



- Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- Range: $(-\infty, \infty)$
- $\displaystyle\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty$
- $\displaystyle\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$

Arctangent

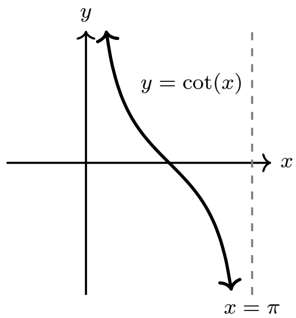


- Domain: $(-\infty, \infty)$
- Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\displaystyle\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$
- $\displaystyle\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

Derivatives:

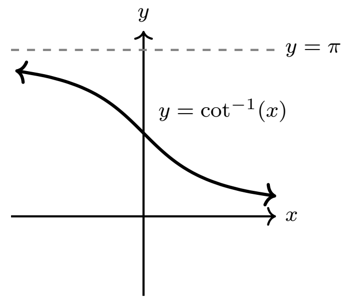
- $\dfrac{d}{dx}(\tan x) = \sec^2 x$
- $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

Restricted cotangent



- Domain: $(0, \pi)$
- Range: $(-\infty, \infty)$
- $\lim\limits_{x \to 0^+} \cot x = \infty$
- $\lim\limits_{x \to \pi^-} \cot x = -\infty$

Arccotangent

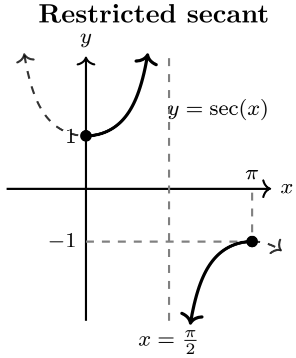


- Domain: $(-\infty, \infty)$
- Range: $(0, \pi)$
- $\lim\limits_{x \to -\infty} \cot^{-1} x = -\pi$
- $\lim\limits_{x \to \infty} \cot^{-1} x = 0$

Derivatives:

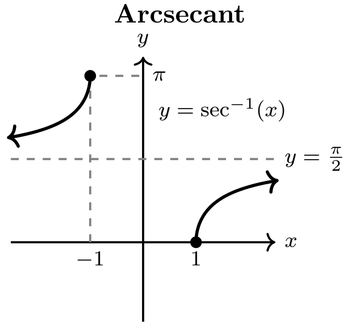
- $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
- $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$

Restricted secant



- Domain: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
- Range: $(-\infty, -1] \cup [1, \infty)$
- Vertical asymptote: $x = \frac{\pi}{2}$

Arcsecant

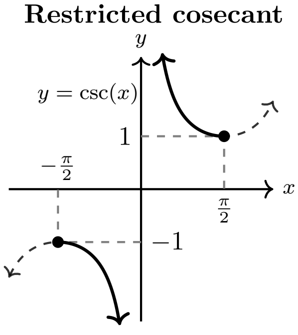


- Domain: $(-\infty, -1] \cup [1, \infty)$
- Range: $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
- Horizontal asymptote: $y = \frac{\pi}{2}$

Derivatives:

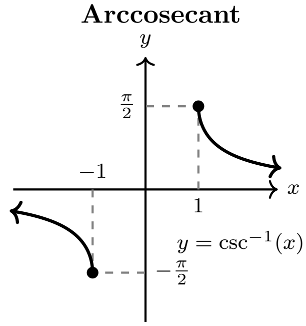
- $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
- $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2-1}}$

Restricted cosecant



- Domain: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
- Range: $(-\infty, -1] \cup [1, \infty)$
- Vertical asymptote: $x = 0$

Arccosecant



- Domain: $(-\infty, -1] \cup [1, \infty)$
- Range: $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
- Horizontal asymptote: $y = 0$

Derivatives:

- $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$
- $\dfrac{d}{dx}(\csc^{-1} x) = -\dfrac{1}{|x|\sqrt{x^2-1}}$

9 Limit Calculations

The following limit examples come **DIRECTLY** from the Math 152 homework for Sections 8.8, 10.1, 10.2, 10.3, 10.5. Practice these limit calculations now, so that you can focus on the concepts in those sections when you get to them.

Exercise 55. $\lim_{x\to\infty} \frac{1}{x^{0.0009}}$

Exercise 56. $\lim_{x\to\infty} \frac{1}{x^{-0.0009}}$

Exercise 57. $\lim_{x\to\infty} \ln|x+1| - \ln|x|$

Exercise 58. $\lim_{x\to\infty} \left(\tan^{-1}\left(e^{2x}\right)\right)^2$

Exercise 59. $\lim_{x\to\infty} \frac{1}{\ln x}$

Exercise 60. $\lim_{x\to 0^+} x^2 \ln x$

Exercise 61. $\lim_{x\to 1^-} \sin^{-1} x$

For the next three problems, try writing out a table of values for $n = 1, 2, 3, 4, 5, 6$ first.

Exercise 62. $\lim_{n\to\infty} 6 + (0.9)^n$

Exercise 63. $\lim_{n\to\infty} 6 + (-0.8)^n$

Exercise 64. $\lim_{n\to\infty} 6 + (-1)^n$

Exercise 65. $\lim_{x\to\infty} e^{-x}$

Exercise 66. $\lim_{n\to\infty} \frac{1-3n^4}{n^4+7n^3}$

Exercise 67. $\lim_{n\to\infty} \frac{(-1)^n}{5n-6}$
(Assume n is an integer, so that $(-1)^n$ makes sense)

Exercise 68. $\lim_{n\to\infty} \frac{-\sin n}{6n}$

Exercise 69. $\lim_{n\to\infty} \frac{n}{11^n}$

Exercise 70. $\lim_{n\to\infty} \frac{\ln(n+5)}{\sqrt{n}}$

For the next two problems, you will use logarithms:

Exercise 71. $\lim_{n\to\infty} \sqrt[n]{n}$

Exercise 72. $\lim_{n\to\infty} \left(1 - \frac{5}{n}\right)^n$

— Factorials

Recall that, for example, $7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 5040$

In other words, $n! = n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1$.

This also means, for example, that $(n+1)! = (n+1)n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1$.

And we have some useful ways to manipulate this:

$$\begin{aligned} n! &= n(n-1)(n-2)(n-3)\ldots 3\cdot 2\cdot 1 \\ &= n\cdot(n-1)! \\ &= n(n-1)(n-2)! \end{aligned}$$

Exercise 73. $\lim_{n\to\infty} \frac{(4n+1)!}{(4n)!}$

Exercise 74. $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right)$

Exercise 75. $\lim_{n\to\infty} \ln(\sqrt{n+1})$

Exercise 76. $\lim_{n\to\infty} \arctan\left(\frac{1}{n+1}\right)$

Exercise 77. $\lim_{n\to\infty} \frac{7^{n+1}n!}{(n+1)!\,7^n}$

Exercise 78. $\lim_{n\to\infty} \left(\frac{6}{(8n+9)^n}\right)^{\frac{1}{n}}$
(Hint: simplify first! No logarithms needed!)

Exercise 79. $\lim_{n\to\infty} \frac{(n+1)^{15}}{15^{n+1}}\frac{15^n}{n^{15}}$

Exercise 80. $\lim_{n\to\infty} \frac{(n+1)^2(n+5)!}{(n+1)!\,6^{3(n+1)}}\frac{n!\,6^{3n}}{n^2(n+4)!}$

Exercise 81. $\lim_{n\to\infty} \frac{((n+1)!)^2}{(2(n+1))!}\frac{(2n)!}{(n!)^2}$

Exercise 82. $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{3n}$

Exercise 83. $\lim_{n\to\infty} \frac{(n+1)!}{19^{n+1}(n+1)^{(n+1)}}\frac{19^n n^n}{n!}$

Exercise 84.

(a) $\lim_{x\to\infty} \frac{(\ln x)}{x}$

(b) $\lim_{x\to\infty} \frac{(\ln x)^2}{x}$

(c) $\lim_{x\to\infty} \frac{(\ln x)^3}{x}$

(d) $\lim_{x\to\infty} \frac{(\ln x)^{100}}{x}$

10 Sigma Notation

Our working definition of the integral involves a very long sum:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\frac{b-a}{n}f\left(a+\frac{b-a}{n}\right)+\frac{b-a}{n}f\left(a+\frac{b-a}{n}2\right)+\ldots+\frac{b-a}{n}f\left(a+\frac{b-a}{n}n\right)\right]$$

The ellipsis (...) in this expression means something like 'and so on' or 'et cetera', and it's fine to use ellipses occasionally. However, we will have long calculations and precise definitions involving sums like this, so it's better if we have a more definite way of writing 'and so on'. The tool we will use is called Sigma notation.[3] Formally,

$$\boxed{\sum_{i=a}^{b} f(i) \text{ means } f(a)+f(a+1)+f(a+2)+f(a+3)+\ldots+f(b)}$$

So for example, we can write:

$$\sum_{i=4}^{7}(\sin(i)+2i^2+1) = [\sin(4)+2(4^2)+1]+[\sin(5)+2(5^2)+1]+[\sin(6)+2(6^2)+1]+[\sin(7)+2(7^2)+1]$$

and our integral above becomes

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{b-a}{n}f\left(a+\frac{b-a}{n}k\right)$$

Exercise 85. Fully simplify $\sum_{i=2}^{5}(i+1)^2$

Exercise 86. Expand, but **do not simplify** $\sum_{i=5}^{8}\cos(i+c)$

Exercise 87. Fill in the blanks: $\sum_{i=1}^{5} i^2+3i = \sum_{i=0}^{4}\underline{\hspace{2cm}} = \sum_{i=2}^{6}\underline{\hspace{2cm}}$

Now let's look at the sum $\sum_{i=6}^{10}(2i+3)$. The value of this sum is 95, but let's look at what a little factoring and rearranging can show us about how these sums work:

$$\begin{aligned}\sum_{i=6}^{10}(2i+3) &= [2(6)+3]+[2(7)+3]+[2(8)+3]+[2(9)+3]+[2(10)+3]\\ &= 2(6)+2(7)+2(8)+2(9)+2(10)+3+3+3+3+3\\ &= 2[6+7+8+9+10]+[3+3+3+3+3]\\ &= 2\sum_{i=6}^{10} i+\sum_{i=6}^{10}3\end{aligned}$$

So now we have these general properties:[4]

$$\boxed{\begin{array}{l}\text{(a)}\ \displaystyle\sum_{i=a}^{b}[f(i)+g(i)] = \sum_{i=a}^{b}f(i)+\sum_{i=a}^{b}g(i)\\[2ex] \text{(b)}\ \displaystyle\sum_{i=a}^{b}[cf(i)] = c\sum_{i=a}^{b}f(i) \quad \text{if c is a constant}\end{array}}$$

Exercise 88. If $\sum_{i=1}^{n} f(i) = 5$ and $\sum_{i=1}^{n} g(i) = 7$, then what is $\sum_{i=1}^{n}[3f(i)+g(i)]$?

[3] The symbol $\sum$ is the capital Greek letter Sigma

[4] These two properties together are called *linearity*. Notice that the limits, derivatives, and integrals have the analogous properties. If you have taken Math 250, this should look familiar.

Next we have some formulas for common sums:

- $\displaystyle\sum_{i=1}^{n} 1 = 1+1+1+\ldots+1 = n$
- $\displaystyle\sum_{i=1}^{n} i = 1+2+3+\ldots+n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$

Exercise 89. Fully simplify $\displaystyle\sum_{i=1}^{200}(3i^2+2i+7)$

Exercise 90. Expand and simplify the individual terms. Leave your answer as a sum. $\displaystyle\sum_{n=0}^{4}\frac{(-1)^{n+1}}{8n-3}$

Exercise 91. Write in sigma notation: $\dfrac{1}{20}+\dfrac{3}{25}+\dfrac{3^2}{30}+\dfrac{3^3}{35}+\dfrac{3^4}{40}$

We can also use sigma notation to write an infinite sum:

$\displaystyle\sum_{i=a}^{\infty} f(i)$ means $f(a)+f(a+1)+f(a+2)+f(a+3)+\ldots$

In Section 10.2, you will analyze infinite sums as the limit of their **partial sums**:
If $\displaystyle\sum_{i=a}^{\infty} f(i)$ is an infinite sum, then its **nth partial sum** is the finite sum $\displaystyle\sum_{i=a}^{a+n-1} f(i)$

For example, the fifth partial sum of $\displaystyle\sum_{i=1}^{\infty}\frac{1}{i}$ is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

Exercise 92. Write out the fourth partial sum of the infinite sum:

(a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{4}{3^n}+\frac{1}{2^n}\right)$ (b) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x-1}{5}\right)^n$

For the next two problems, begin by writing out the first three terms of the sum, and then looking for a pattern.

Exercise 93. Using sigma notation, express the repeating decimal as an infinite sum: $0.\overline{12}$.

Exercise 94. Start with a square whose area is 144. Create a second square by joining the midpoints of the sides of the first square. Repeat this process. Use sigma notation to write out the infinite sum of the sums of the areas of these squares.

Exercise 95. Write a formula for the nth partial sum of $\displaystyle\sum_{i=1}^{\infty}\frac{10}{i}-\frac{10}{i+1}$
(This is called a **telescoping** sum because the middle terms will collapse like an old-timey telescope)