

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination
January 2026

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let $f(t)$ and $g(t)$ be nonzero polynomials with integer coefficients without any common complex roots. Prove that there exist polynomials $u(t)$ and $v(t)$ with integer coefficients so that

$$u(t)f(t) + v(t)g(t)$$

is a nonzero integer.

2. Let A be an associative commutative ring with 1. Prove that an element $f = a_0 + a_1x + \dots + a_nx^n$ of the polynomial ring $A[x]$ is nilpotent if and only if all a_i are nilpotent elements of A .
3. (a) Let G be a group of order $1025 = 41 \cdot 25$. Prove that G has an abelian normal subgroup H such that G/H is abelian.
(b) Prove that there exists a nonabelian group G of order 1025.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let A be an $n \times n$ -invertible matrix with entries in the field $\mathbb{Q}$ of rational numbers. Suppose that $A^4 - 6A = 3A^{-1}$. Show that $5|n$, and that such a matrix A is unique up to similarity in $M_{n \times n}(\mathbb{Q})$.
5. Let A be an $n \times n$ matrix with complex coefficients. Prove that there exist matrices B and C such that all of the following properties hold:
 - $A = B + C$;
 - $BC = CB$;
 - B is diagonalizable;
 - $C^n = 0$.

End of Session on Algebra

RUTGERS UNIVERSITY
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January 2026

Session 1. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. For $a > 0$ find

$$I_a = \int_0^\infty \frac{x \sin x}{x^2 + a^2} dx.$$

2. Let Φ be the family of all analytic functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

on the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, such that $|a_n| \leq n$ for each $n \in \mathbb{N}$. Prove that for any sequence $(f_n)_{n \in \mathbb{N}} \subseteq \Phi$ there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}} \subseteq \Phi$ that converges uniformly on any compact set $K \subseteq D$ to an analytic function F on D .

3. Let $P(z)$ be a monic polynomial of degree $n \geq 1$, i.e.

$$P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0.$$

Then either $P(z) = z^n$ or there exists $z_0 \in \mathbb{C}$ such that $|z_0| = 1$ and

$$|P(z_0)| > 1.$$

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Show that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0.$$

5. Let $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ denote an open disc with radius $r > 0$ centered at $z_0 \in \mathbb{C}$. Let $R = D(1, 1) \cap D(i, 1)$. Construct a finite number $n \in \mathbb{N}$ of conformal maps $\varphi_1, \dots, \varphi_n$ such that

$$F = \varphi_1 \circ \dots \circ \varphi_n$$

maps R conformally to $D(0, 1)$.

End of Session 1

RUTGERS UNIVERSITY
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Written Qualifying Examination

Session on Real Variables and Elementary Point-Set Topology, January 2026

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology, January 2026. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. For any two μ -measurable functions $f, g : X \rightarrow \mathbb{C}$, let

$$d(f, g) := \int \frac{|f - g|}{1 + |f - g|} d\mu. \quad (1)$$

(a) Show that d is a metric on the space of μ -measurable functions on X (provided we identify functions that differ on a null set, as usual).

(b) Show that if $\{f_n\}$ is a sequence of μ -measurable functions on X and f is μ -measurable on X , then f_n converges to f in measure if and only if $\lim_{n \rightarrow \infty} d(f_n, f) = 0$.

(c) Now, suppose μ is only σ -finite, and not necessarily finite. Consider a sequence $\{f_n\}$ of L^1 functions and f is another L^1 function. Is the following statement true: f_n converges to f in measure if and only if $\lim_{n \rightarrow \infty} d(f_n, f) = 0$. If so, prove it, if not, provide a counterexample.

SOLUTION: (a) It is obvious that, $d(f, g) = d(g, f)$ and that $d(f, f) = 0$. Now, if $d(f, g) = 0$, then

$$\frac{|f - g|}{1 + |f - g|} = 0 \quad (2)$$

almost everywhere, and so $f = g$ almost everywhere. Finally, the triangular inequality follows from the fact that $x \mapsto x/(1 + x)$ is monotone increasing and subadditive on $\mathbb{R}^+$: indeed

$$\frac{a + b}{1 + a + b} \leq \frac{a}{1 + a} + \frac{b}{1 + b} \quad (3)$$

is equivalent to

$$(a + b)(1 + a)(1 + b) \leq a(1 + a + b)(1 + b) + b(1 + a + b)(1 + a) \quad (4)$$

which is equivalent to

$$ab(a + b + 2) \geq 0. \quad (5)$$

Thus, for $x, y, z \in \mathbb{C}$,

$$\frac{|x - z|}{1 + |x - z|} \leq \frac{|x - y| + |y - z|}{1 + |x - y| + |y - z|} \leq \frac{|x - y|}{1 + |x - y|} + \frac{|y - z|}{1 + |y - z|} \quad (6)$$

(b) Suppose that f_n converges to f in measure, and for any $\epsilon > 0$, let

$$E_n(\epsilon) := \{x : |f_n(x) - f(x)| > \epsilon\}. \quad (7)$$

We have

$$d(f_n, f) = \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|} d\mu + \int_{X \setminus E_n} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \quad (8)$$

which we bound using the fact that $x \mapsto \frac{x}{1+x} < 1$ for $x \in \mathbb{R}_+$, as

$$d(f_n, f) \leq \mu(E_n) + \mu(X) \frac{\epsilon}{1 + \epsilon} \quad (9)$$

and so

$$\lim_{n \rightarrow \infty} d(f_n, f) \leq \mu(X) \frac{\epsilon}{1 + \epsilon}. \quad (10)$$

Since this holds for all ϵ ,

$$\lim_{n \rightarrow \infty} d(f_n, f) = 0. \quad (11)$$

Conversely, if $\lim_{n \rightarrow \infty} d(f_n, f) = 0$, we have

$$d(f_n, f) \geq \int_{E_n} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \geq \mu(E_n) \frac{\epsilon}{1 + \epsilon} \quad (12)$$

so $\lim \mu(E_n(\epsilon)) = 0$ for any ϵ , and so f_n converges to f in measure.

(c) If μ is only σ -finite, then part (b) does not hold: consider $X = \mathbb{R}$, $\mathcal{M} = \mathcal{L}$ and μ is the Lebesgue measure. This space is σ -finite ($\mathbb{R} = \bigcup_{i \in \mathbb{Z}} [i, i+1)$). Consider $f_n = \frac{1}{n} \chi([0, n])$, which converges to 0 in measure, however,

$$d(f_n, 0) = \int \frac{|f_n(x)|}{1 + |f_n(x)|} dx = n \frac{\frac{1}{n}}{1 + \frac{1}{n}} = \frac{1}{1 + \frac{1}{n}} \quad (13)$$

which does not tend to 0. (The converse is still true.)

2. Let m be the Lebesgue measure on $\mathbb{R}$ endowed with the σ -algebra $\mathcal{L}$ of Lebesgue-measurable subsets of $\mathbb{R}$. Recall that m is a complete measure.

(a) Is the product measure $m \times m$ on the product σ -algebra $\mathcal{L} \otimes \mathcal{L}$ complete? If so, prove it, if not, provide a counterexample.

Hint: Show that if $A \in \mathcal{L}$, $A \neq \emptyset$, and $B \notin \mathcal{L}$, then $A \times B \notin \mathcal{L} \otimes \mathcal{L}$.

(b) Consider the set $E := \{(x, y) \in \mathbb{R}^2 : \frac{1}{2} < x < 1, 0 < y < \frac{1}{\sqrt{1+x^2}}\}$. Prove that E is $m \times m$ -measurable, and that $m \times m(E) = \arcsin(1) - \arcsin(\frac{1}{2})$ (make sure to justify your answers!).

Hint: To prove that E is $m \times m$ -measurable, you may use the fact that Borel-measurable sets of $\mathbb{R}^2$ are $m \times m$ -measurable.

SOLUTION: (a) We first show that if $A \in \mathcal{L}$, $A \neq \emptyset$, and $B \notin \mathcal{L}$, then $A \times B \notin \mathcal{L} \otimes \mathcal{L}$. Let $x \in A$. We have

$$B = (A \times B)_x \quad (14)$$

and so, if $A \times B$ were measurable, then B would be measurable. Thus $A \times B$ is not measurable.

Now, consider $A = \{0\}$ and $B \in \mathcal{P}([0, 1]) \setminus \mathcal{L}$ (such a set exists, for instance the set N from section 1.1 in Folland). Since A is measurable and non-empty, $A \times B$ is not measurable, but $A \times B \subset \{0\} \times [0, 1]$ whose measure is 0. Therefore, $m \times m$ is not complete.

(b) We have that

$$E = g^{-1}((\frac{1}{2}, 1) \times (0, 1)) \quad (15)$$

where the function $g : (x, y) \mapsto (x, y\sqrt{1+x^2})$ is continuous. Since $(\frac{1}{2}, 1) \times (0, 1)$ is a rectangle, E is Borel-measurable, thus $m \times m$ -measurable. By Tonelli's theorem

$$m \times m(E) = \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{\sqrt{1+x^2}}} dy dx = \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{1+x^2}} dx = \arcsin(1) - \arcsin(\frac{1}{2}). \quad (16)$$

3. Suppose that $f_n \rightarrow f$ pointwise on X and there exists some $g \in L^p(X)$ such that $|f_n(x)| \leq g(x)$ for all $x \in X$ and that a sequence $h_n \in L^{p'}(X)$ converges to h in $L^{p'}(X)$, where p' is the conjugate exponent of p , prove that $\int_X f_n h_n \rightarrow \int_X f h$.

SOLUTION: First, we have $|f(x)| \leq g(x)$ for all $x \in X$. Next, by Hölder's inequality, $g(x)|h(x)| \in L^1(X)$, so $\int_X f_n h_n, \int_X f h$, as well as $\int_X f_n h$, are convergent integrals in $L^1(X)$. Furthermore, $|f_n(x) - f(x)||h(x)| \leq 2g(x)|h(x)|$, so we can apply DCT to conclude that $\int_X |f_n(x) - f(x)||h(x)| \rightarrow 0$ as $n \rightarrow \infty$, and that

$$\left| \int_X f_n h - \int_X f h \right| \leq \int_X |f_n(x) - f(x)||h(x)| \rightarrow 0,$$

from which we conclude that $\int_X f_n h \rightarrow \int_X f h$. Finally,

$$\left| \int_X f_n h_n - \int_X f h \right| \leq \int_X |f_n| |h_n - h| + \int_X |f_n(x) - f(x)||h(x)|$$

and

$$\int_X |f_n| |h_n - h| \leq \left(\int_X |f_n|^p \right)^{1/p} \left(\int_X |h_n - h|^{p'} \right)^{1/p'} \leq \left(\int_X |f|^p \right)^{1/p} \left(\int_X |h_n - h|^{p'} \right)^{1/p'} \rightarrow 0$$

using $\|h_n - h\|_{L^{p'}(X)} \rightarrow 0$. We thus conclude that $\int_X f_n h_n \rightarrow \int_X f h$.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Suppose $\mathcal{H}$ is a real Hilbert space. Let $\mathcal{K} \subset \mathcal{H}$ be a non-empty closed convex subset. Recall that a set $\mathcal{K}$ in a vector space is convex if for any $f, g \in \mathcal{K}$ and any $c \in [0, 1]$, $cf + (1 - c)g \in \mathcal{K}$. Let $x \in \mathcal{H}$. Then define

$$L := \inf_{k \in \mathcal{K}} \|x - k\| \quad (17)$$

- a) Show that the infimum in (??) is attained at a unique point $k(x) \in \mathcal{K}$.
b) Let $\mathcal{K}_1 = \text{span} \{ \sum_{i=1}^n c_i u_i : \| \sum_{i=1}^n c_i u_i \|_{L^2[0,1]} \leq 1 \}$, where $u_1, \dots, u_n \in L^2[0, 1]$ (which you may take to be linearly independent or even orthonormal), and $\mathcal{K}_2 = \{ v \in C[0, 1] : \int_0^1 v(x)^2 dx \leq 1 \}$ considered as subsets of $L^2[0, 1]$. Verify that both are convex subsets. Are they closed subsets of $L^2[0, 1]$? Justify your answer.

SOLUTION: a). Assume without loss of generality that $x \notin \mathcal{K}$. Take a sequence $k_i \in \mathcal{K}$ with $\|k_i - x\| \rightarrow L$. Recall for any $a, b \in \mathcal{H}$ the parallelogram law gives

$$2(\|a\|^2 + \|b\|^2) = \|a - b\|^2 + \|a + b\|^2. \quad (18)$$

Setting $a = x - k_i$ and $b = x - k_j$ in (??) gives

$$2(\|x - k_j\|^2 + \|x - k_i\|^2) = \|2x - (k_i + k_j)\|^2 + \|k_i - k_j\|^2. \quad (19)$$

By convexity of $\mathcal{K}$, $\|2x - (k_i + k_j)\|^2 \geq 4L^2$. Thus

$$\|k_i - k_j\|^2 \leq 2(\|x - k_j\|^2 + \|x - k_i\|^2) - 4L^2, \quad (20)$$

implying $\{k_i\}$ is a Cauchy sequence. By completeness, there is a limit k_∞ and by closedness of $\mathcal{K}$, the limit is contained in $\mathcal{K}$. Moreover, $\|x - k_\infty\| = \lim_{i \rightarrow \infty} \|x - k_i\| = L$. Thus the infimum distance is obtained. For uniqueness, observe that if k_∞ and k'_∞ satisfy $\|x - k'_\infty\| = \|x - k_\infty\| = L$, then by the parallelogram law again and convexity of $\mathcal{K}$

$$2(L^2 + L^2) \geq \|2x - (k'_\infty + k_\infty)\|^2 + \|k'_\infty - k_\infty\|^2 \geq 4L^2 + \|k'_\infty - k_\infty\|^2, \quad (21)$$

whence $k_\infty = k'_\infty$.

For b), both $\mathcal{K}_1$ and $\mathcal{K}_2$ are characterized by the property $\|u\|_{L^2[0,1]} \leq 1$. Take any $f, g \in \mathcal{K}_1$ and any $c \in (0, 1)$, by the triangle inequality, we have

$$\|cf + (1 - c)g\|_{L^2[0,1]} \leq c\|f\|_{L^2[0,1]} + (1 - c)\|g\|_{L^2[0,1]} \leq c + 1 - c = 1,$$

which shows that $cf + (1 - c)g \in \mathcal{K}_1$. The same proof works for $\mathcal{K}_2$.

$\mathcal{K}_2$ is not a closed subset of $L^2[0, 1]$, because the space $C[0, 1]$ is not complete under the $L^2[0, 1]$ and there are sequences $v_n \in \mathcal{K}_2$ converging to some v in $L^2[0, 1]$, which is no longer in $C[0, 1]$, therefore $v \notin \mathcal{K}_2$.

$\mathcal{K}_1$ is a closed subset of $L^2[0, 1]$ because any two norms on a finite dimensional space are equivalent so we could use $\sum_{i=1}^n |c_i|$ as a norm for $\sum_{i=1}^n c_i u_i$, so if any sequence f_j in $\mathcal{K}_1$ is converging in $L^2[0, 1]$ to some $f \in L^2[0, 1]$, then the coefficients of f_j would also converge, which shows that $f \in \mathcal{K}_1$.

5. Suppose $\mathcal{B}$ is a Banach space. A bounded linear map $A : \mathcal{B} \rightarrow \mathcal{B}$ is called *compact* if whenever a sequence $\{x_i\}_{i=1}^\infty \in \mathcal{B}$ satisfies $\|x_i\| \leq C$ for some $C > 0$, the sequence $\{A(x_i)\}_{i=1}^\infty$ has a convergent subsequence. Let $K(x, y) \in C([0, 1]^2)$ and any function $f \in C([0, 1])$ is endowed with the norm $\|f\| := \sup_{x \in [0, 1]} |f(x)|$. Show that the linear map

$$A : C([0, 1]) \rightarrow C([0, 1]) \quad (22)$$

given by

$$Af(y) := \int_0^1 K(x, y)f(x)dx \quad (23)$$

is a bounded and compact operator, where Af is also endowed with the norm $\|Af\| := \sup_{y \in [0, 1]} |Af(y)|$.

SOLUTION: Note that for each $y \in [0, 1]$,

$$|Af(y)| \leq \int_0^1 |K(x, y)f(x)|dx \leq \sup_{x \in [0, 1]} |K(x, y)| \cdot \sup_{x \in [0, 1]} |f(x)|. \quad (24)$$

Thus

$$\sup_{y \in [0, 1]} |Af(y)| \leq K \sup_{x \in [0, 1]} |f(x)|, \quad (25)$$

where

$$K := \sup_{(x, y) \in I^2} |K(x, y)|. \quad (26)$$

Thus the map A is bounded. To show compactness, let us assume $f_i \in C([0, 1])$ satisfies

$$\sup_{x \in [0, 1]} |f_i(x)| \leq C. \quad (27)$$

We claim the family $\{A(f_i)\}_{i=1}^{\infty}$ is equicontinuous. Thus by the Arzela-Ascoli theorem, the sequence $\{A(f_i)\}_{i=1}^{\infty}$ thus has a convergent subsequence which suffices to show the operator A is compact.

Let us prove the equicontinuity. Note that

$$Af_i(y) - Af_i(y') = \int_0^1 (K(x, y) - K(x, y')) f_i(x) dx, \quad (28)$$

so that

$$|Af_i(y) - Af_i(y')| \leq \int_0^1 |(K(x, y) - K(x, y'))| |f_i(x)| dx \leq C \int_0^1 |(K(x, y) - K(x, y'))| dx \quad (29)$$

The uniform continuity of K on $[0, 1]^2$ gives that for all $\eta > 0$ there exists a $\delta = \delta(\eta) > 0$ so that whenever $|y - y'| \leq \delta$ there holds $|K(x, y) - K(x, y')| \leq \eta$. Fix $\varepsilon > 0$ and set $\delta = \delta(\varepsilon/C)$. Then we get for all $|y - y'| \leq \delta$ that

$$|Af_i(y) - Af_i(y')| \leq C \cdot \frac{\varepsilon}{C} = \varepsilon. \quad (30)$$

Since δ does not depend on i , this establishes the desired equicontinuity.