

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination
January 2025

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Prove that the alternating group A_6 has no subgroup of index 5.
2. Classify all linear transformations T of $V = \mathbb{R}^2$ satisfying $T^2 = I$.
3. Let p be a positive prime number. Let R be the ring of Gaussian integers ($R = \{a + bi \in \mathbb{C} : a, b \in \mathbb{Z}\}$). Prove that pR is a prime ideal in R if and only if $p \equiv 3 \pmod{4}$.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

- 4.** Prove that any group of order $2555 = 5 \cdot 7 \cdot 73$ is cyclic.
- 5.** Let R be a commutative ring with unit. If M is an R -module, and $f : M \rightarrow R^2$ is a surjective homomorphism, show that $M \cong N \oplus R^2$ for some module N .

End of Session on Algebra

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Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

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Part I. Answer all questions.

Problem 1: Evaluate

$$\int_0^{\infty} \frac{\log x}{x^{\frac{1}{2}}(x+1)^2} dx.$$

Problem 2: Let D be a bounded domain in $\mathbb{C}$. Suppose $f_n : D \rightarrow \mathbb{C}$ is a sequence of holomorphic functions satisfying

$$\int_D |f_n(z) - f(z)| dz d\bar{z} = 0$$

for a continuous function f on D . Prove that f is holomorphic on D .

Problem 3: Construct $w = f(z)$ that maps conformally the region

$$U = \{z : |z| < 1, \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$$

on the region $\{w : |w| < 1\}$.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

Problem 1: Let

$$f_n(z) = \sum_{k=0}^n \frac{1}{k!z^k}.$$

Prove that for sufficiently large $n \in \mathbb{Z}^+$ and sufficiently large $R > 1$, all zeros of $f_n(z)$ are in

$$\{z : |z| < R\}.$$

Problem 2: Suppose f is analytic on $\overline{D_r} = \{z : |z| \leq r\}$ and f has no zeros in $\overline{D_r}$. Prove that

$$|f(0)| = \lim_{n \rightarrow \infty} \left(f(r) f(r\omega_n) f(r\omega_n^2) \dots f(r\omega_n^{n-1}) \right)^{1/n},$$

where $\omega_n = e^{2\pi i/n}$.

End of Session on Complex Variables and Advanced Calculus

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Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

Problem 1. Let f, g be absolutely continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and $h = f \circ g$ be their composition.

Prove that if g is nondecreasing, then h is absolutely continuous.

Problem 2.

Let dx denote Lebesgue measure on the Lebesgue σ -algebra of $\mathbb{R}$. Assume $f \in L^1(\mathbb{R}, dx)$ is real-valued. Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) := \int_{\mathbb{R}} f(y) e^{-(x-y)^2} dy.$$

Prove that the map $x \mapsto g(x)$ is Lipschitz continuous on $\mathbb{R}$.

Problem 3.

Let $X = Y = [0, 1]$, both equipped with the Borel σ -algebra $\mathcal{B}([0, 1])$. Let λ denote Lebesgue measure restricted to $\mathcal{B}$. Let ν denote the counting measure: for $A \subset [0, 1]$, and $\mathcal{N} \subset A$ finite, $\nu(A) = \sup_{\mathcal{N} \subset A} \{\text{number of elements in } \mathcal{N}\}$. Let $D := \{(x, y) \in X \times Y : x = y\}$. Let χ_D denote the characteristic function of the set D , which may be seen as a function of x with y a parameter, or a function of y with x a parameter, or a function of (x, y) — depending on the context.

Show that the three integrals

$$I_1 := \int_Y \left(\int_X \chi_D(x) d\lambda(x) \right) d\nu(y)$$

$$I_2 := \int_X \left(\int_Y \chi_D(y) d\nu(y) \right) d\lambda(x)$$

$$I_3 := \int_{X \times Y} \chi_D(x, y) d(\lambda(x) \times \nu(y))$$

yield three different results, and explain (briefly) why this is not a counterexample to the Fubini-Tonelli theorem.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

Problem 4.

Let dx denote Lebesgue measure on the Lebesgue σ -algebra of $\mathbb{R}$.

Let $\{f_n : \mathbb{R} \rightarrow \mathbb{C}; n \in \mathbb{N}\}$ be a sequence of functions with uniformly bounded $L^2(\mathbb{R}, dx)$ -norm. Suppose that there is an $f \in L^2$ such that for any $h \in L^2$ one has that $\int_{\mathbb{R}} f_n(x)h(x)dx \rightarrow \int_{\mathbb{R}} f(x)h(x)dx$ when $n \rightarrow \infty$. Suppose furthermore that the sequence $\{g_n : \mathbb{R} \rightarrow \mathbb{C}; n \in \mathbb{N}\} \subset L^2(\mathbb{R}, dx)$ converges in L^2 topology to g .

Show that $\int_{\mathbb{R}} f_n(x)g_n(x)dx \rightarrow \int_{\mathbb{R}} f(x)g(x)dx$ as $n \rightarrow \infty$.

(HINT: The stated assumptions do **not** imply that $f_n \rightarrow f$ in L^2 topology.)

Problem 5.

Let $X = [0, 1]$, and let dx denote Lebesgue measure on X .

(a) Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be the periodic odd function of period 2 that takes the value 1 for $x \in (0, 1)$ and the value 0 at $x = 1$. (Note that this completely defines F .) Now consider the sequence $\{f_n; n \in \mathbb{N}\}$ with $f_n(x) := F(2^{n-1}x)$ restricted to X .

Show that when $n \rightarrow \infty$, then $\int_X g(x)f_n(x)dx \rightarrow 0$ for each $g \in L^2(X, dx)$, but that $f_n \not\rightarrow 0$ in $L^2(X, dx)$ topology, and that $f_n \not\rightarrow 0$ pointwise a.e. in X , and that $f_n \not\rightarrow 0$ in measure either.

(HINT: It helps to graph the first few members of the sequence $\{f_n\}$, say for $n \in \{1, 2, 3\}$. It also helps to recall Bessel's inequality.)

(b) Consider the sequence $\{f_n; n \in \mathbb{N}\}$, with $f_n(x) := n\chi_{[0, 1/n)}(x)$.

Show that when $n \rightarrow \infty$, then $f_n \rightarrow 0$ pointwise a.e. in X , and $f_n \rightarrow 0$ also in measure, but $f_n \not\rightarrow 0$ pointwise in X , and also $f_n \not\rightarrow 0$ weakly in $L^2(X, dx)$. (Hint: For the last one of these four tasks, it suffices to find a suitable $g \in L^2(X, dx)$ such that $\int_X n\chi_{[0, 1/n)}(x)g(x)dx \not\rightarrow 0$ when $n \rightarrow \infty$.)

End of Session on Real Variables and Elementary Point-Set Topology