

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination
August 2023

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Classify the groups of order $2023 = 7 \times 17^2$ up to isomorphism. (You may use without proof the well-known result that if p is a prime, then every group of order p^2 is abelian.)
2. Let $\mathbb{R}[x, y]$ be the polynomial ring over $\mathbb{R}$ in the variables x, y and let I be the principal ideal generated by $f(x, y) = x^2 + y^2 + 1$. Prove that the ring $R = \mathbb{R}[x, y]/I$ has infinitely many maximal ideals.
3. Let $A = \mathbb{Z} \oplus \mathbb{Z}$ be the free abelian group of rank 2. Compute the number of subgroups $B \subseteq A$ of index 3.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

For the following questions, recall that an element r of a ring R is said to be *nilpotent* if there exists a positive integer k such that $r^k = 0$.

4.

- (i) [7 pts] Prove that if N is a nilpotent $n \times n$ matrix over $\mathbb{C}$ and I is the $n \times n$ identity matrix, then there exists an $n \times n$ matrix A over $\mathbb{C}$ such that $A^2 = I + N$.
- (ii) [3 pts] Prove that there does not exist a 2×2 matrix B over the field $\mathbb{F}_2$ with 2 elements such that

$$B^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- 5.** Let R be an associative commutative ring with identity 1. Prove that an element $f(z) = a + bz$ of the polynomial ring $R[z]$ is a unit if and only if a is a unit in R and b is nilpotent in R .

End of Session on Algebra

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Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

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Part I. Answer all questions.

- 1.** Use Green's theorem to evaluate the integral

$$\int_C \sqrt{1 + e^{x^2}} dx + 4xy dy,$$

where C is the boundary of the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$, with the standard counter-clockwise orientation.

- 2.** Assume $\xi > 0$ and compute

$$\int_{\mathbb{R}} \frac{\cos(2\pi x \xi)}{x^2 + 1} dx.$$

- 3.** Does there exist a holomorphic surjection from the open unit disk $\mathbb{D}$ to the whole complex plane $\mathbb{C}$? If so, provide one; if not, prove that it does not exist.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let $z_1, z_2, \dots, z_n$ be points on the unit circle in the complex plane. Prove that there exists a point z on the unit circle such that

$$\prod_{i=1}^n |z - z_i| = 1.$$

5. Let $A = \{1 < |z| < 2\}$ and $B = \{1 < |z| < 3\}$. Show that there is no holomorphic function $f : A \rightarrow B$ such that f extends continuously to the closure $\overline{A}$ to $\overline{B}$ and $f(\{|z| = 1\}) \subset \{|z| = 1\}$, $f(\{|z| = 2\}) \subset \{|z| = 3\}$.

End of Session on Complex Variables and Advanced Calculus

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Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

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Part I. Answer all questions.

1. Let f_n for $n \in \mathbb{N}$ and f be complex-valued measurable functions on a measure space (X, μ) and suppose that f_n converges to f in measure. Then some subsequence of f_n converges to f pointwise μ -a.e.

2. Let $a_n \geq 0$ for $n \in \mathbb{N}$, and let $0 < p < q$.

(i) Prove that

$$\left(\sum_{n=1}^{\infty} a_n^q \right)^{1/q} \leq \left(\sum_{n=1}^{\infty} a_n^p \right)^{1/p}.$$

(ii) Prove that for every $N \in \mathbb{N}$ we have

$$\left(\sum_{n=1}^N a_n^p \right)^{1/p} \leq N^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{n=1}^N a_n^q \right)^{1/q}.$$

3. Show that if $f(x, y) = ye^{-(1+x^2)y^2}$ for each $x, y \in \mathbb{R}$, then

$$\int_0^{\infty} \left(\int_0^{\infty} f(x, y) dx \right) dy = \int_0^{\infty} \left(\int_0^{\infty} f(x, y) dy \right) dx.$$

Use the preceding equality to prove the formula

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. Let (X, μ) be a finite measure space. For every pair of measurable functions $f, g : X \rightarrow \mathbb{C}$ let

$$d(f, g) = \int_X \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} d\mu(x).$$

- (i) Show that d is a metric on the space of equivalence classes of measurable functions which differ only on measure zero sets.
 - (ii) Show that a sequence of measurable functions $(f_n)_{n \in \mathbb{N}}$ converges in measure to a measurable function f if and only if $\lim_{n \rightarrow \infty} d(f_n, f) = 0$.
 - (iii) Can one drop the assumption that $\mu(X) < \infty$ in (i)?
5. Let $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$ denote the Euclidean ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$, where $|x| = (\sum_{j=1}^d x_j^2)^{1/2}$ is the standard Euclidean norm for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. For a Lebesgue integrable function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ we define

$$A_r f(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy,$$

where m denotes the d -dimensional Lebesgue measure on $\mathbb{R}^d$. Show that $A_r f(x)$ is jointly continuous in $r > 0$ and $x \in \mathbb{R}^d$, and using this deduce that the set

$$\{x \in \mathbb{R}^d : \sup_{r>0} |A_r f(x)| > \lambda\}$$

is open in $\mathbb{R}^d$ for every $\lambda > 0$.

End of Session on Real Variables and Elementary Point-Set Topology