Oral Qualifying Exam Syllabus

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I. PDE and Analysis

A. Partial Differential Equations

i. Laplace's Equation

- Fundamental solutions including derivations
- Green's identities
- Mean Value Theorem
- Green functions and the Poisson kernel, including the Dirichlet problem on the unit ball
- Uniqueness of solutions to the Dirichlet problem via the Maximum Principle
- Perron's existence method for the Dirichlet problem

ii. Wave Equation

- Characteristic surfaces (light cones)
- Energy estimates
- C^2 uniqueness via energy
- Fundamental solutions (derivation for $n = 1$ and $n = 3$)
- Duhamel's Principle
- Dispersive (L^∞) estimate

B. Analysis

i. Hilbert Spaces

- Closed subspace decomposition
- Self-duality
- Bessel's identity
- Completeness

ii. Elements of Fourier Analysis

- Schwarz space as a Fréchet space
- Properties of convolutions and Young's Inequality

- Approximations of the identity (density of C_c^∞ in L^p and the C^∞ Urysohn Lemma)
- Fourier Transform on $\mathbb{R}^n$:
  - Properties
  - Riemann-Lebesgue Lemma
  - Fourier inversion
  - Plancherel Formula

iii. **Theory of Distributions**

- Definition of convergence of test functions and continuity of functionals
- Derivatives and convolutions of distributions and test functions
- Density of C_c^∞ in $\mathcal{D}'$
- Tempered distributions - examples and Fourier Transform

iv. **L^2-Sobolev Spaces**

- Definition of H_s
- Equivalent norms
- Properties, including density of H_s in H_t $(t < s)$
- Sobolev embedding theorem for H_s
- Rellich's compactness theorem
- Definition of a localized H_s space
- The elliptic regularity theorem for constant coefficient operators

v. **Inequalities**

- Hölder's Inequality
- Minkowski's Integral Inequality
- Sobolev Inequality applied to the hydrogen atom (stability of matter)
- Heisenberg's Inequality (Uncertainty Principle)

II. Mathematical Physics

A. Newtonian Point Mechanics

i. Newton's equation of motion

- Galilean invariance
- The Lorentz Force law for test particles (test particle motion in a uniform electric and magnetic field)

ii. Lagrangian formulation

- Euler-Lagrange equations
- Equivalence to Newton's equation of motion
- The 2-body problem for attractive/repulsive $1/r$ potential
- Noether's Theorem (symmetry and conservation laws)

iii. Hamiltonian formulation

- Legendre transformations
- Equivalence to Newton's equation of motion

B. Einsteinian Point Mechanics

i. Changes with respect to Classical Physics

- Poincaré Group
- 4-vectors and invariants (proper time, wave operator etc.)

ii. Maxwell's equations for fields given charges/currents

- Derivation of wave equation in Lorentz gauge
- Gauge invariance
- Covariant form of Maxwell's equations

C. Quantum Mechanics

i. Schrödinger's Equation

- The (non-relativistic) hydrogen atom as a 1-body problem (Pauli's wave equation)
- N-body Schrödinger Equation
- Dirac Equation

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