Oral Qualifying Exam Syllabus

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Schemes

1. **Sheaves**: Presheaves and associated sheaves, morphisms of sheaves, stalks, pull-back and push-forward
2. **Schemes**: Spec and Proj, morphisms of schemes, generic points, varieties
3. **First properties of schemes**: Reduced, integral, noetherian, normal schemes, morphisms of finite type, finite morphisms, open and closed immersions, fibre products
4. **Separated and proper morphisms**: Definition of separated and proper morphisms, valuative criteria, scheme-theoretic images
5. **Sheaves of Modules**: Definition of $\mathcal{O}_X$-modules, quasi-coherent and coherent sheaves, constructions of $\mathcal{O}_X$-modules, invertible sheaves, vector bundles, twisted sheaves
6. **Divisors**: Weil divisors, Cartier divisors, Weil and Cartier divisor class groups, equivalence for locally factorial schemes, Picard groups
7. **Projective morphisms**: Definition, criteria and characterization of projective morphisms, ample and very ample invertible sheaves, linear systems, blowing ups
8. **Differentials**: Derivations, sheaves of relative differentials, connection to nonsingularity, (co)normal sheaves, canonical sheaves, geometric genus

Cohomology Theory of Sheaves

1. **Derived functors**: Abelian categories, complexes, derived functors, δ-functors
2. **Cohomology of sheaves**: The category of sheaves of $\mathcal{O}_X$-modules has enough injectives, Grothendieck's vanishing theorem

3. **Cohomology of Noetherian Affine Schemes**: Cohomological characterization of noetherian affine schemes
4. **Čech Cohomology**: Definition, computation of sheaf cohomology groups for a noetherian separated scheme
5. **Cohomology of Projective Spaces**: Properties of cohomology groups of projective schemes, criterion for ampleness
6. **Extension Groups and Sheaves**
7. **Serre Duality Theorem**: Duality for $\mathbf{P}_k^n$, dualizing sheaves, duality for projective schemes

Curves

1. **Nonsingular Curves**
2. **Divisors on Curves**
3. **Riemann-Roch**
4. **Hurwitz' Theorem**
5. **Embeddings in Projective Space**
6. **Elliptic Curves**: The j-invariant, the group structure, the Jacobian variety, Elliptic functions, the Hasse invariant, rational points on an elliptic curve

The Geometry of Flag Varieties (of Type A)

1. **Grassmannians and the Plucker Embedding**
2. **Flag Varieties, Bruhat Decomposition and Schubert Varieties**
3. **Schubert Classes, Kleiman's Transversality, and Intersections**
4. **Line Bundles and Characters, Borel-Weil**
5. **Singularities of Schubert Varieties**: Normality, rationality of singularities, cohomology of line bundles
6. **The Grothendieck Ring and a Degeneration of the Diagonal of a Flag Variety**
7. **Positivity of the Structure Constants of the Grothendieck Ring**

Linear Algebraic Groups (over Algebraically Closed Fields)

1. **Definitions and First Properties**: Definition of algebraic groups and functor of points, morphisms, G-spaces, Jordan Decomposition, Representations
2. **Commutative Algebraic Groups**
3. **Lie Algebras**
4. **Homogeneous Spaces and Quotients**
5. **Parabolic Subgroups, Borel Subgroups, Solvable Groups**
6. **Classification of Reductive Groups**

References

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[2] R. Vakil, *The Rising Sea: Foundations of Algebraic Geometry*, November 18, 2017 draft.

[3] M. Brion, *Lectures on the Geometry of Flag Varieties*, 2003.

[4] T.A. Springer, *Linear Algebraic Groups (2nd ed.)*, Birkhauser, 1998.

[5] J.S. Milne, *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*, Cambridge University Press, 2017.