Oral qualifying exam syllabus for Daniel Tan

Fall 2023

Major Topic: Vertex operator algebras

1. Definitions and properties
   (a) Formal calculus.
   (b) The notions of vertex algebra and of vertex operator algebra, and basic properties.
   (c) Equivalent definitions of vertex operator algebras.
2. Modules for vertex operator algebras
   (a) The notion of a module for a vertex operator algebra.
   (b) Equivalent definitions for modules.
   (c) Construction theorems for vertex algebras and modules.
3. Examples of vertex algebras and their modules
   (a) Heisenberg vertex operator algebras.
   (b) Lattice vertex operator algebras.
   (c) Affine vertex operator algebras.
   (d) Virasoro vertex operator algebras.
4. Intertwining operators and characters
   (a) The notions of intertwining operators and intertwining maps.
   (b) The notion of $P(z)$-tensor products.
   (c) Examples of intertwining operators and $P(z)$ tensor products: Heisenberg, lattice, affine and Virasoro.
   (d) Statements of the convergence, associativity and commutativity of intertwining operators.
   (e) The notions of the q-trace and characters of modules.
   (f) Examples of characters: Heisenberg, lattice, affine and Virasoro.
   (g) Convergence statements on characters.
   (h) The statement of the modular invariance of the space of characters.

Minor Topic: Infinite dimensional Lie algebras

1. Basic Lie theory
   (a) The finite-dimensional representation theory of $\mathfrak{sl}(2)$.
   (b) Universal enveloping algebras and the Poincaré-Birkhoff-Witt theorem.
   (c) Free Lie algebras.
2. Kac-Moody Lie algebras
   (a) The definition and construction of Kac-Moody Lie algebras.
   (b) The Weyl group.
   (c) Standard modules.
   (d) The Weyl-Kac character formula.
3. Examples of infinite dimensional Lie algebras
   (a) $A_1^{(1)}$ and $A_2^{(2)}$, their root systems and their Weyl groups.
   (b) The denominator formula for $A_1^{(1)}$ and the Jacobi triple product identity.
   (c) (Twisted) $\widehat{\mathfrak{sl}_2}$ and its use to prove the Rogers-Ramanujan identities.

References

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