Syllabus for Oral Qualifying Exam

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1 Major Topic: Partial Differential Equations

1.1 Laplace's equation and the heat equation

- Fundamental solution, the Poisson kernel and the heat kernel.
- Mean value formulas and Liouville's theorem for C^2 solutions.
- Weak and strong maximum principles for C^2 solutions.
- Harnack's inequality and Hopf's lemma for C^2 solutions.
- Energy methods and uniqueness.

1.2 Sobolev spaces

- Weak derivatives for locally integrable functions.
- Interior and global approximation by smooth functions.
- Extension and trace operators for C^1 domains.
- Gagliardo–Nirenberg inequality, Sobolev embeddings and compactness.
- Poincaré's inequality.

1.3 Second order elliptic equations

- Weak formulations for operators in divergence form.
- Existence and uniqueness in H_0^1: Lax–Milgram theorem and the Fredholm alternative.
- Energy estimates and regularity.
- Maximum principles for operators in nondivergence form.

1.4 Stationary Navier–Stokes equations

- Divergence and vorticity.
- Helmholtz decomposition for $W^{1,2}$ vector fields.
- Euler equations and Bernoulli's law in 2D.
- Stokes problem and Leray's problem in 2D.

2 Minor Topic: Functional Analysis

2.1 Banach spaces and linear operators

- Hahn–Banach theorem: analytic and geometric forms.
- Baire's category theorem and the uniform boundedness principle.
- Open mapping theorem and closed graph theorem.
- Adjoint operator and orthogonality.

2.2 Weak topologies

- Motivation: a coarser topology has more compact sets.
- Weak and weak* topology: Banach–Alaoglu theorem.
- Reflexive spaces: bidual and the canonical injection.
- Separability and metrizability: compactness and sequentially compactness.

2.3 Hilbert spaces

- Cauchy–Schwarz inequality and the parallelogram law.
- Duality: Riesz representation theorem.
- Hilbert basis and orthogonal projection.

2.4 Compact operators

- Approximation by finite rank operators.
- Riesz–Fredholm theory: Fredholm alternative.
- Spectral decomposition of self-adjoint compact operators.

2.5 Nonlinear analysis

- Fréchet and Gâteaux derivatives in Banach spaces.
- Sub-supersolution method in cones.
- Sard's theorem and Brouwer's degree in $\mathbb{R}^n$.
- Completely continuous operators and Leray–Schauder degree.
- Fixed point theorems.

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