Oral Qualifying Exam Syllabus

GUANHUA SHAO

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1 Major: Minimal surfaces theory and Riemannian Geometry

1. Basic Riemannian Geometry

   - Geodesics: Gauss Lemma, Minimizing properties, First variation formulas
   - Curvature: Sectional, Ricci, Scalar curvature, Mean curvature and Gauss curvature
   - Jacobi fields, Conjugate points
   - Complete manifolds: Theorem of Hadamard, Hopf-Rinow Theorem
   - Comparison theorem: Rauch Comparison Theorem
   - Variation of Energy: the Theorem of Bonnet-Myers, Theorem of Synge

2. Variational formulas and consequences

   - Definition and minimal surface equation in $\mathbb{R}^n$
   – First Variation Formula and Monotonicity Formula, Convex Hull Property
   – The Theorem of Bernstein
   – The Strong Maximum Principle
   – Second Variation Formula, Morse Index, and Stability
   - Fisher-Colbre-Schoen Theorem and Pogorelov Inequality, Criterion for Stability

3. Curvature Estimates and Compactness Theorem

   - Basic Compactness Theorem, Proof of a special case of Allard's Theorem by Brian White
   - Small Energy Curvature Estimates
   – Hersch's technique for the Laplacian operator on spheres
   – Minimal Cone

4. Minimal Surfaces in Three-Manifolds

   – Compactness Theorems with A Priori Bounds by Choi and Schoen

5. Existence results

   - Douglas & Rado theorem for Plateau Problem

2 Minor: Elliptic Partial Differential Equations

1. Laplacian operator and Poisson equations

   - Mean value property, Maximum principles
   - Fundamental solution and Green's representation
   - Green's function and Poisson's Formula, Smoothness, Analyticity, Gradient estimate, Convergence theorem, Harnack inequality, Liouville Theorem

2. Sobolev Spaces

   - Definitions and Elementary properties
   - Approximations, Chain Rule
   - Trace and Extentions
   - Sobolev Embedding Theorems, Rellich-Kondrachov Compact embedding Theorem, Poincare inequalities

3. Weak Solutions for Divergence Form second order Elliptic Operators

   - Lax Milgram Theorem, Fredholm Alternative, L^2 Regularity

References

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[3] L. C. Evans (2010) *Partial Differential Equations, 2nd Edition*, American Mathematical Society.

[4] D. Gilbarg, N.S.Trudinger (1998) *Elliptic Partial Differential Equations of Second Order*, Springer

[5] D. Ketover(2023) *Lecture Notes in Topics in Geometry*

[6] H.I. Choi, R. Schoen (1985) *The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature, Invent. math, 81, 387-394*