1 Descriptive Set Theory

Polish Spaces

- Definitions, examples, and basic theorems
  - Alexandrov's Theorem
  - A nonempty $C \subseteq \mathbb{N}^{\mathbb{N}}$ is closed iff $C = [T]$ for some pruned $T \subseteq \mathbb{N}^{<\mathbb{N}}$.
  - A nonempty $K \subseteq \mathbb{N}^{\mathbb{N}}$ is compact iff $C = [T]$ for some finitely branching pruned $T \subseteq \mathbb{N}^{<\mathbb{N}}$.
  - $\mathbb{N}^{\mathbb{N}}$ is not K_σ.
- Universal Polish Spaces and Perfect Polish Spaces
  - Examples of universal Polish spaces ($\mathbb{H} = [0,1]^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$).
  - Cantor-Bendixon Theorem
  - Every uncountable Polish space contains a homeomorphic copy of $2^{\mathbb{N}}$.
  - Every nonempty Polish space is the continuous image of $\mathbb{N}^{\mathbb{N}}$.

Baire Category Theorem and the Baire Property

- Baire Category Theorem
- Borel subsets of a Polish space are Baire measurable.
- If $A \in \mathrm{BP}(X)$ then A is meager or comeager in some nonempty open set. If X is Baire, exactly one holds.
- Kuratowski-Ulam
- No nonempty perfect polish space admits a Baire measurable well-order.

The Borel hierarchy

- Existence of $2^{\mathbb{N}}$-universal sets for $\boldsymbol{\Sigma}^0_\alpha$ and $\boldsymbol{\Pi}^0_\alpha$
- Change of topology
- Perfect set property for Borel subsets of Polish spaces
- Standard Borel spaces

Analytic Sets

- Analytic sets are the continuous image of $\mathbb{N}^{\mathbb{N}}$.
- Existence of analytic but not Borel sets
- f is a Borel function iff $\mathrm{graph}(f)$ is Borel iff $\mathrm{graph}(f)$ is analytic.

Closure Properties and Uniformization

- Countable-to-one Borel images of Borel sets are Borel.
- Luzin-Novikov
- Any two Polish spaces of the same cardinality are Borel isomorphic.

Equivalence Relations

- An analytic equivalence relation has a Borel selector iff it has an analytic transversal.
- Smooth equivalence relations

  - If E admits a Borel selector, E is smooth (and counterexample to the converse).
  - E is smooth iff E has a Borel separating family.
  - Generic ergodicity (definitions, examples, obstruction to smoothness, example of $\chi_B \neq \chi$)

- CBERs
  - A CBER E is smooth iff the quotient X/E is a standard Borel space.
  - A CBER E is smooth iff E admits a Borel (equivalently, analytic) transversal iff E admits a Borel selector.
  - Feldman-Moore
  - Hyperfinite Equivalence Relations
  - If E is a smooth CBER, then E is hyperfinite.
  - Closure properties of hyperfinite and smooth equivalence relations.

2 Set Theory

Ordinals and Cardinals

- Basic definitions (regular, singular, limit, strong limit, cofinality)
- Kőnig's Lemma ($\kappa^{cf(\kappa)} > \kappa$)

Clubs and Stationary Sets

- Intersection of fewer than κ clubs in κ is a club.
- Diagonal intersection of κ-many clubs in κ is a club.
- (Fodor's lemma) Regressive functions are constant on a stationary set.
- Stationary reflection

L

- Con(ZF) $\implies$ Con($V = L$)
- $L \models GCH$
- There are no measurable cardinals in L.

Trees

- Aronszajn trees (definition, existence)
- Characterization of weakly compact cardinals with tree property (κ is weakly compact iff it's inaccessible and has the tree property)

Forcing

- Δ-system Lemma
- ccc posets preserve cardinals.
- Add forcing
  - Add(ω, λ) has ccc.
  - Independence of CH
- Product Forcing
  - Product lemma (mutual genericity)
- Martin's Axiom
  - MA$_{\aleph_1} \implies \neg \exists$ Suslin tree