Oral Examination

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1 Topics in Probability on metric spaces

1. Measures in Metric Spaces, tightness
2. Weak Convergence of measures
3. Prohorov's theorem and relative compactness in spaces of probability measures
4. Measures on function space $C[0,1]$ existence of Wiener measure, Brownian motion
5. Conditional Probabilities, conditional expectations, disintegration of measures

2 Topics in Stochastic Processes

1. Filtrations, stopping times, stopped σ-algebra.
2. Martingales, submartingales and supermartingales, examples
3. Martingale convergence theorems: optional stopping theorem, Doobs martingale inequalities
4. Uniformly integrable martingales, convergence theorems.
5. Kolmogorov-Centsov existence theory of stochastic processes

3 Optimal Transport

1. Basic formulation and set up of optimal transportation problems.
   (i) Monge Problem
   (ii) Kantorovitch relaxation
   (iii) Some examples where solution to Monge and Kantorovitch coincide, where they do not as well.
2. Kantorovitch duality
   (i) Concepts from convex analysis (Fenchel Rockafellar duality, Legendre transformation)
   (ii) Kantorovitch duality

3. Optimal Transportation for Quadratic Costs
    (i) Knott-Smith Optimality Criterion
    (ii) Brenier's Theorem
    (iii) Generalizations of Brenier to strictly convex functions (Gangbo and McCann, etc)

3.1 Optimal Transportation Metrics

(i) Wasserstein distances and Monge-Kantorovitch distances

(ii) Wasserstein distances and weak convergence in probability

(iii) Connections to statistical mechanics (Tanaka Theorem)

4 Large deviations

(i) Why/What are large deviations?
    (a) Large deviations for Bernoulli random variables, relation to DeMoivre-Laplace
    (b) Large deviations for normal random variables
    (c) Heuristic arguments for large deviations in Brownian motion paths

(ii) Formulation of a Large Deviation Principle (LDP)
    (a) Motivation for rate function through entropy/energy considerations, statistical mechanics
    (b) Rate functions
    (c) Equivalence to Varadhan integral lemma

(iii) LDP in Topological Vector Spaces

(iv) Cramér's Theorem in Polish spaces, particular cases in $\mathbb{R}^d$

(v) Sanov's Theorem for probabilities on Polish spaces

(vi) Generalizations of Sanov's result to Markov chains, through regularity assumptions such as ergodicity of the Markov chain

5 Applications to Statistical Mechanics

(i) Connections of large deviations to statistical mechanics equilibrium (Lanford)

(ii) Tanaka's theorem, trend to equilibrium in solutions to Boltzmann equation

(iii) Vlasov equation uniqueness proof (Dobrushin)

References

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[8] Karatzas, Ioannis and Shreve, Steven, *Brownian Motion and Stochastic Calculus*, Springer, 1991

[9] Lanford, O.E. , *Entropy and equilibrium states in classical statistical mechanics*, Springer Verlag, Lecture Notes in Physics Vol. 20., 1973