Syllabus for Oral Qualifying

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1 Major Topic: Partial Differential Equations

1.1 Classical PDEs

- Laplace's equation. Fundamental solution in $\mathbb{R}^n$. Green's function for bounded domains. Newtonian potential.
- Harmonic functions. Mean value theorem. Weak and strong maximum principle. Liouville's theorem. Hopf's lemma.
- Heat equation and wave equation.

1.2 Sobolev Spaces

- Weak derivative. Extension of Sobolev functions. Approximation by smooth functions.
- Gagliardo–Nirenberg–Sobolev inequality. Rellich compact embedding.
- Poincaré's inequality. Trace Operator. Hardy's inequality.

1.3 Second Order Elliptic Equation

- Existence and uniqueness of solutions to linear elliptic PDE. Lax–Milgram theorem. Fredholm's Alternative.
- Regularity. H^2 estimates of weak solutions.
- Maximum principle of general second order elliptic pde.
- De Giorgi–Nash–Moser theory, L^∞ estimates.
- Moving plane method.
- Leray–Schauder fixed point theorem.
- Stationary Navier–Stokes problem, Leray's argument *reductio ad absurdum*. Existence for flux zero boundary value problem.
- Invading domain method for stationary Navier–Stokes problems in exterior domains. Existence of D-solutions.

2 Minor Topic: Functional Analysis

2.1 Classical Theory

- Banach spaces. Hilbert spaces. Ascoli–Arzelà theorem.
- Hahn–Banach theorem. Open mapping theorem. Uniform boundedness principle. Closed graph theorem.
- Riesz representation theorem. Lax–Milgram theorem.
- Weak and weak* topology. Banach–Alaoglu–Bourbaki compactness theorem.
- Spectrum of self-adjoint compact operators.

2.2 Distribution and Fourier Analysis

- Distribution space $\mathcal{D}'$. Topology of $C_c^\infty(\Omega)$.
- Schwartz space $\mathcal{S}$, tempered distribution $\mathcal{S}'$.
- Fourier transform in L^1, L^2, $\mathcal{S}$, $\mathcal{S}'$. Plancherel theorem
- Fundamental solutions of PDEs with constant coefficients. Application to the Stokes problem.

References

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