Oral Exam Syllabus

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- Major Topic: Functional Analysis and Operator Algebras

  1. (Hausdorff) Topological Vector Spaces
     (a) Locally Convex Spaces: seminorms, weak topologies metrizability, Fréchet spaces
     (b) Han-Banach Theorem(s)
     (c) Krein-Milman Theorem.
  2. Hilbert Spaces
     (a) Elementary properties: inner products, orthonormal bases, Parseval's Identity, polarization identities, Riez-Representation theorem.
     (b) Bounded Linear maps: Normal maps, Self-adjoint maps, Unitary maps.
  3. Compact Operators on Hilbert Space
     (a) Elementary Properties: Density of Finite Rank operators,
     (b) When Self-adjoint:Spectrum, Spectral Theorem, Functional Calculus
     (c) Fredholm Alternative
     (d) Trace class and Hilbert-Schmidt operators
  4. Banach spaces
     (a) Bounded linear maps, Compact linear maps, Dual map
     (b) Baire Category Theorem, Uniform Boundedness Principle, Open Mapping Theorem, Closed Graph Theorem
     (c) Closed Subspaces, Quotient spaces, quotient maps, Riesz Lemma
     (d) Dual spaces, Reflexivity, Sequential Compactness, Weak* toplogy, Banach-Alaoglu Theorem
     (e) Separable spaces, Uniformly Convexity, Mazur's Theorem.
  5. C^* Algebras
     (a) Examples and Elementary properties, properties of involution and self-adjoint elements
     (b) Commutative C^* algebras, characters and Gelfand-Naumark Theorem.
     (c) Spectral Theorem, Spectral Mapping Theorem, and Functional Calculus

   (d) Positivity
   (e) Approximate Identities and C^* unitization
   (f) Russo-Dye theorem, C^*-involutions are isometries
   (g) Homomorphisms of C^* algebras

6. Concrete C^* algebras.
   (a) Strong, weak, Strong*, σ-Strong, σ-Strong*, σ-weak topologies: definition, continuity of natural operations and continuous linear functionals
   (b) Baire Functional Calculus for Normal operators on Hilbert Space
   (c) Polar Decompostion
   (d) Vigier's Theorem, Commutants, Double Commutant Theorem, Kaplansky Density Theorem
   (e) Von Neumann algebras as dual spaces
   (f) Representations of C^* algebras: states, pure states, GNS construction

References: W. Rudin. *Functional Analysis*
H. Brezis. *Functional Analysis, Sobolev Spaces, and Partial Differential Equations.*
Professor's Notes

- Minor Topic: Function Spaces for PDEs

1. $L^p(\Omega)$ spaces, $\Omega \subset \mathbb{R}^n$
   (a) Examples and Elementary Properties
   (b) Reflexivity, Separability, Duality
   (c) Convolution and approximation
   (d) Compactness in L^p, $1 < p < \infty$
2. Sobolev Spaces
   (a) Definition, examples
   (b) Absolute continuity in 1D, Differentiability on lines, products, compositions, change of variables
   (c) Separability, Duality
   (d) $W^{1,\infty}$: Lipschitz functions, Radamacher's Theorem, Extensions of Lipschitz functions
   (e) Convolution and Approximation
   (f) Extensions for smooth or half-space
   (g) Gagliardo-Nirenberg, Morrey, and Poincaré inequalities
   (h) Embedding Theorem(s) and Compactness
3. Sobolev Spaces $H^s(\mathbb{R}^n)$
   (a) Definitions, equivalent norms, inner product
   (b) Density of embedding H^t in H^s
   (c) Sobolev Embedding for H^s

   (d) Rellich compactness

4. Distributions
   (a) Test Functions and Distributions on an open set, supports convolutions, density of test functions
   (b) Schartz space and Tempered Distributions, Fourier transform, convolutions, density of test functions

5. Laplace/Poisson Equation
   (a) Fundamental Solution
   (b) Properties of Harmonic Functions: Mean value properties, Harnack's inequality
   (c) Maximum Principle
   (d) Regularity of Solutions

6. Convex Analysis
   (a) Projection onto a Convex set in Hilbert Space, Stampacchia and Lax-Milgram Theorems
   (b) Conjugate convex functions, Young's inequality

References: L. Evans and R. Gariepy. *Measure Theory and Fine Properties of Functions.*
H. Brezis. *Functional Analysis, Sobolev Spaces, and Partial Differential Equations.*,
E. Lieb and M. Loss. *Analysis.*, Chapters 6, 9, 10.