Oral Qualifying Exam Syllabus

Theodore Gonzales

Major Topic: Algebraic Geometry [2]

The majority of the major topic is from Vakil[2], but Hartshorne should be okay. The order of the syllabus strongly follows that of the sections of Vakil's table of contents[2].

Preliminaries

1. Definitions of Categories, Functors, Limits, Adjoints. When do functors preserve limits and colimits?

2. Definitions and constructions of presheaves, sheafification, stalks, sheaves on a base, $\mathscr{O}_X$-modules, direct and inverse image sheaf.

Schemes

1. Definition/existence of Spec as left adjoint of Γ. Scheme structure on $\mathrm{Spec}(A)$, Construction of $\mathrm{Proj}(S)$.

2. Properties of schemes: Reducedness, integrality, affine communication lemma [2] (5.3), normality and factoriality.

3. Definition of quasicoherent sheaves and characterization in terms of distinguished affine base, along with their properties: finite type, finitely presented, and coherent. Associated points and zero divisors extended to quasicoherent sheaves.

Morphisms

1. Definitions of morphisms of locally ringed spaces, morphisms of schemes, functoriality of Proj.

2. Finiteness conditions on morphisms (finite type, locally of finite type, finite...). Statement and applications of Chevalley's Theorem and fundamental theorem of elimination theory (aka $\mathbb{P}^n$ is compact) [2](8.4).

3. Definition of closed embeddings, closed scheme-theoretic image [2](9.4.4), Cartier divisors.

4. Construction of fibered products (either by hand or through representable functors) and interpretations as fibers of morphisms or base changes. Properties preserved by base change. Definition and construction of normalization.

5. Definitions and properties of separated and proper morphisms. Reduced to separated theorem [2](11.4).

Geometric Properties

1. Definition of dimension. Statement of Noether normalization, Krull height theorem and applications to dimension. Statement about dimensions of fibers of morphisms of varieties (12.4.1)[2].

2. Definition of Zariski tangent space and relation to regularity, smoothness, and Jacobian matrix at k valued points. Algebraic Hartogs' Lemma [2](13.5) and applications to divisors.

Bundles

1. Vector bundles as spaces vs as locally free finite rank sheaves. Quasicoherent sheaf on $\text{Proj}(S)$ associated to a graded module.

2. Definition/existence of $\mathbb{P}^n_A$ as representable functor. Line bundles on $\mathbb{P}^n_k$.

3. Differences between Weil divisors, Cartier divisors, and invertible ideal sheaves (and when they are the same).

4. Projective morphisms (Hartshorne definition). Definition of relative Spec and Proj and relationship to total space of locally free finite rank sheaf.

5. Properties of ample and very ample line bundles. Grassmannian as representable functor. Proof that locally representable functors are representable [2]10.1.7.

Cohomology and Such

1. Definitions of sheaf cohomology via Čech and derived functors. Statement/proof of Serre vanishing. Using cohomology of projective space (e.g., to find degree of subscheme).

2. Statement of Riemann Roch for line bundles on curves (or proof assuming canonical sheaf is dualizing). Difference between arithmetic and geometric genus. Statement/use of Serre duality for line bundles.

Minor Topic: Tautological Classes of Matroids [1]

1. Statement of localization theorems/integration formula for the permutohedral variety X_E.

2. Construction of line bundles on X_E associated to generalized permutohedra and construction of tautological classes associated to matroids (Section 3).

3. Main steps in the proof of the Tutte polynomial formula for tautological classes (Section 4).

4. How to get Bergman classes from Tautological classes (Section 7).

5. Main steps in proof of log concavity of Tutte polynomial in the realizable case (Section 9).

References

[1] Andrew Berget, Christopher Eur, Hunter Spink, and Dennis Tseng. Tautological classes of matroids. *Inventiones mathematicae*, 233(2):951–1039, May 2023. doi: 10.1007/s00222-023-01194-5.

[2] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry Notes (most recent version)*. URL https://math.stanford.edu/~vakil/216blog/.