

Topics for oral qualifying exam for Jishen Du

Fall, 2022

Major topic: Vertex operator algebras

The general theory of vertex operator algebras and modules, basic examples and some basic material on intertwining operators and tensor products, as presented in [FLM] (Chapters 1–8), [FHL], [H1], [H2] and [LL]. The specific topics are:

1. Definitions and properties.
 - (a) Formal calculus.
 - (b) The notions of vertex algebra and of vertex operator algebra, in terms of the duality properties (rationality, commutativity and associativity) and in terms of the Jacobi identity.
 - (c) Basic properties, including the weak commutativity, weak associativity, commutator formula, associator formula, operator product expansion.
 - (d) Conformal elements.
 - (e) General construction theorem for vertex operator algebras.
2. Examples of vertex (operator) algebras.
 - (a) Vertex (operator) algebras based on Heisenberg Lie algebras.
 - (b) Vertex (operator) algebras based on affine Lie algebras.
 - (c) Vertex (operator) algebras on the Virasoro algebra.
 - (d) Vertex (operator) algebras on even lattices.
3. Representations of vertex (operator) algebras.
 - (a) The notions of module and its variants (generalized module, lower-bounded generalized module, grading-restricted generalized module), in terms of the duality properties (rationality, commutativity and associativity) and in terms of the Jacobi identity.
 - (b) Basic properties, including the weak commutativity, weak associativity, commutator formula, associator formula, operator product expansion.
 - (c) General construction theorems for (generalized) modules for vertex operator algebras.
 - (d) Examples of modules for the vertex operator algebras listed above.
4. Intertwining operators and tensor products of modules

- (a) The notions of (logarithmic) intertwining operator, in terms of the duality properties (rationality, commutativity and associativity) and in terms of the Jacobi identity, fusion rules.
- (b) The statements of the convergence, associativity and commutativity for (logarithmic) intertwining operators, (logarithmic) operator product expansion. (Proofs are not necessary.)
- (c) The statement of the convergence for the (pseudo-) q -traces of products of (logarithmic) intertwining operators. (Proofs are not necessary.)
- (d) The geometry of vertex operators and intertwining operators in terms of $P(z)$, the sphere with punctures $0, z, \infty$ and standard local coordinates.
- (e) The definition of $P(z)$ -tensor product of (generalized) modules, basic properties.

Minor topic: semisimple Lie algebras and its representation theory

The general theory of finite dimensional complex semisimple Lie algebras and its representation theory, as presented in [Hum1] and [Hum2].

1. Basics

- (a) Engel's theorem and Lie's theorem.
- (b) Killing form and Cartan's Criterion.
- (c) Weyl's theorem for complete reducibility.
- (d) Representation theory of $\mathfrak{sl}_2(\mathbb{C})$.

2. Classification of semisimple Lie algebras

- (a) Root space decomposition of semisimple Lie algebra.
- (b) (Abstract) root system, Weyl group, and its Classification.
- (c) Isomorphism theorem, Serre relation, existence theorem. (i.e. correspondence between Lie algebras and root systems.)
- (d) PBW theorem, Free Lie algebra.

3. Representation theory

- (a) Basics of weight, highest weight module, Verma module, and so on.
- (b) Results for finite dimensional modules.
- (c) Characters, Weyl's formula and dimension formula.
- (d) Harish-Chandra's theorem.
- (e) BGG Category $\mathcal{O}$, Verma's theorem, BGG theorem, BGG Reciprocity.

References

- [FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On Axiomatic Approaches to Vertex Operator Algebras and Modules, *Memoirs Amer. Math. Soc.* 104 (1993).
- [FLM] I. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, 1988.
- [H1] Y.-Z. Huang, Lecture notes on vertex algebras and quantum vertex algebras, pdf file.
- [H2] Y.-Z. Huang, Lecture notes on the representation theory of vertex operator algebras and conformal field theory, pdf file.
- [LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Progress in Math., Vol. 227, Birkhäuser, Boston, 2003.
- [Hum1] J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer New York, 2012
- [Hum2] J. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category O* , American Mathematical Society, 2008.