Oral Qualifying Exam Syllabus

Leonidas Daskalakis

November 2021

1 Major Topic: Ergodic Theory

Measure-Preserving Transformations on Probabilty Spaces:

- Measure Preserving Systems (Definition and Examples)
- Poincaré Recurrence
- Ergodicity
- Von Neumann's Ergodic Theorem
- Birkhoff's Ergodic Theorem using Garcia's proof of the Maximal Inequality
- Birkhoff's Ergodic Theorem using the Transference Principle
- Strong-Mixing and Weak-Mixing
- Induced Transformations, Kac's and Kakutaki-Rokhlin's Theorems

Applications to Continued Fractions

- Elementary properties of Continued Fractions
- The Continued Fraction Map and the Gauss Measure
- Badly Approximable Numbers
- Lagrange's Theorem on Periodic Continued Fractions
- Invertible Extension of the Continued Fraction Map

Invariant Measures for Continuous Maps

- Existence of Invariant Measures for Continuous Maps (Krylov–Bogolyubov)
- The structure of T-Invariant Measures (where T is a continuous map on a compact metric space) and the Ergodic Decomposition.
- Unique Ergodicity

- Equidistribution and Weyl's Criterion
- Furstenberg's proof of the Equidistribution for Irrational Polynomials

A sketch of Furstenberg's Proof of Szemerédi's Theorem

- Multiple Recurrence
- Reduction to the Ergodic Case
- Furstenberg's Correspondence Principle
- Multiple Recurrence for Kronecker Systems and Weak-Mixing Systems

2 Minor Topic: Harmonic Analysis

- Interpolation Theory: Distribution function, $L^{p,\infty}$ space, Marcinkiewicz Interpolation Theorem (diagonal), Riesz-Thorin Interpolation Theorem, Young's convolution inequality.
- Fourier Series: Riemann-Lebesgue lemma, Riemann localization principle, Dini and Jordan criteria for pointwise convergence, applications of the uniform boundedness principle, Dirichlet and Fejér kernels, L^p convergence for approximate identities, decay of Fourier coefficients, Gibbs phenomenon.
- Fourier Transforms: Basic properties, Schwartz class and tempered distributions, Parseval and Plancherel theorems, Hausdorff-Young inequality for $1 \leq p \leq 2$, Poisson summation formula, oscillatory integrals, sublevel set estimates and van der Corput's lemma.
- The Hardy-Littlewood maximal function: Vitali Covering lemma, weak $(1,1)$ and strong (p,p) inequalities for $p > 1$, Lebesgue differentiation theorem, control of other maximal operators, strong maximal function, dyadic maximal function, Calderón-Zygmund decomposition.
- The Hilbert Transform: Poisson and conjugate kernels, weak $(1,1)$ and strong (p,p) theorems of Kolmogorov and M. Riesz, characterization of $Hf \in L^1$ for $f \in S$, maximal function, truncated integrals and pointwise a.e. convergence, L^p convergence of Fourier series and integrals for $1 < p < \infty$ in one dimension, convergence in measure for $p = 1$.
- Singular Integrals: The Fourier transform of the kernel, Method of rotations, maximal function, Riesz transforms, Calderón- Zygmund theorem, Hörmander and gradient conditions, non-convolution type, vector-valued extension.
- H1 and BMO: Atoms, sharp maximal function, interpolation for $T : L^\infty \to$ BMO, good-λ inequalities, John-Nirenberg inequality

- Littlewood-Paley Theory: Dyadic block decomposition, square function, Hörmander multiplier theorem, Marcinkiewicz multiplier theorem, spherical maximal function.

References:

- *Ergodic Theory with a view towards Number Theory* by M. Einsiedler and T. Ward.
- *Classical Fourier Analysis* by Loukas Grafakos
- *Fourier Analysis* by Javier Duoandikoetxea