Oral Qualifying Exam Syllabus

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1 Major Topic: Hamilton's Ricci Flow

- Fundamentals of the Ricci Flow
  - Evolution of curvature quantities under flow;
  - Short time existence with DeTurck's trick;
  - Ricci Flow as a gradient flow via Perelman's F-functional and a gauge transformation;
  - Perelman's W-functional and its monotonicity formula;
  - Proof of the No Local Collapsing theorem.
- Maximum Principle and Applications
  - Scalar maximum principle for heat-type equations on closed manifold;
  - Tensor bundle maximum principle, and version for symmetric 2-tensors.
- Ricci Flow on 3-Manifolds
  - Geometric evolution equations in 3-dimensions;
  - Hamilton's 1982 result on closed 3-manifolds with positive Ricci curvature;
  - Sketch of the proof of Hamilton's 1982 result and the derivation of gradient bounds on scalar curvature therein;
  - Curvature pinching estimates and their preservation/improvement under flow.
  - Definition of the normalized Ricci flow;
- Ricci Solitons
  - Definition and canonical form for gradient Ricci solitons, and some basic identities;
  - Definitions of expanding, steady, and shrinking gradient Ricci solitons;
  - Gaussian and Cylinder solitons;
  - Hamilton's cigar soliton, and ruling it out as a potential singularity model in dimension 3 via no local collapsing theorem.

Reference: *Hamilton's Ricci Flow* by Bennett Chow, Peng Lu, and Lei Ni.

2 Minor Topic: Riemannian Geometry

- Fundamentals
  - Definition of smooth manifold and a Riemannian metric;
  - Definitions and basic properties of Riemannian immersions and submersions;
  - Musical isomorphisms, induced fiber metric on tensor bundles, Riemannian volume form;
  - Gradient, divergence, and Laplace-Beltrami operators;
  - Riemannian distance function, equivalence of manifold topology and topology induced from Riemannian metric.
- Connections
  - Affine connection on tangent bundle, locality/tensoriality properties, covariant derivatives of general tensor fields, Levi-Cevita connection on Riemannian manifold, Koszul formula, local coordinate expressions;
  - Definition of Parallel transport and how to recover connection from parallel transport;
- Geodesics
  - Definition of geodesics, local uniqueness and existence;
  - First variation formula, geodesics as critical points of energy functional;
  - Exponential map and normal coordinates.
  - Statement and proof of the Gauss lemma, proof that geodesics locally minimize length, definition of injectivity radius;
  - Statement and proof of the Hopf-Rinow theorem.
- Curvature
  - Definition of curvature tensor, local coordinate expression;
  - Symmetries of the curvature tensor, including algebraic and differential Bianchi identities;
  - Ricci identities for commutation of covariant derivatives;
  - Ricci tensor and scalar curvature, contracted Bianchi identities;
  - Schur's lemma, definition of Einstein manifold.
- Jacobi Fields
  - The Jacobi equation and local existence and uniqueness of Jacobi fields;
  - Jacobi fields in constant-curvature spaces;
  - Definition of conjugate points;
  - Second variation formula for length functional, definition of cut locus, and proof that geodesics do not minimize past conjugate points.
- Other major theorems
  - Gauss-Bonnet theorem;

  - Cartan-Hadamard theorem;
  - Bonnet-Myers theorem;
  - Synge's theorem;

Reference:

- *Introduction to Riemannian Manifolds* by John Lee;
- *Riemannian Geometry* by Manfredo do Carmo.