# Existence, uniqueness, and regularity results for elliptic equations with drift terms in critical weak spaces

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We consider Dirichlet problems for linear elliptic equations of second order in divergence form on a bounded or exterior smooth domain $\Omega$ in $\mathbb{R}^n$, $n \geq 3$, with drifts $\mathbf{b}$ in the critical weak $L^n$-space $L^{n,\infty}(\Omega;\mathbb{R}^n)$, and $\operatorname{div}\mathbf{b} \geq 0$ in $L^{n/2,\infty}(\Omega)$. We first establish existence and uniqueness of weak solutions in $W^{1,p}(\Omega)$ or $D^{1,p}(\Omega)$ for any $p$ with $n' = n/(n-1) < p < n$. By duality, a similar result also holds for the dual problem. Next, we prove $W^{1,n+\epsilon}$ or $W^{2,n/2+\delta}$-regularity of weak solutions of the dual problem for some $\epsilon, \delta > 0$ when the domain $\Omega$ is bounded. By duality, these results enable us to obtain a quite general uniqueness result as well as an existence result for weak solutions belonging to $\bigcap_{p<n'} W^{1,p}(\Omega)$. Finally, we prove a uniqueness result for exterior problems, which implies in particular that (very weak) solutions are unique in both $L^{n/(n-2),\infty}(\Omega)$ and $L^{n,\infty}(\Omega)$. This is a joint work with Hyunseok Kim.