1. For each part, write your answer on the provided line. You are not required to show work, but you may use the provided space for scratch work. For each part, there is no partial credit.

5 pts
(a) Find the slope of the tangent line to the curve $x^3 - y^3 = y - 1$ at the point $(1, 1)$.

5 pts
(b) The total revenue from selling $x$ units of a certain product is $R(x) = 40 - \frac{200}{x + 5}$. Using marginal analysis, estimate the revenue from selling the 6th unit.

5 pts
(c) If $x$ units are produced, then the total cost is $C(x) = x^3 + 4x^2 + 60x + 200$ and the selling price per unit is $p(x) = 100 - 3x$. Find the level of production that maximizes the total profit.

5 pts
(d) Use a linear approximation to estimate the value of $(16.32)^{1/4}$.

5 pts
(e) Calculate the derivative of $f(x) = x^x$. Your final answer must contain only $x$.

5 pts
(f) Find the equation of each horizontal asymptote of $f(x) = \frac{2e^x - 5}{3e^x + 2}$. Write “NONE” as your answer if appropriate.

Solution
(a) Implicitly differentiate the equation with respect to $x$ to obtain

$$3x^2 - 3y^2 \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

Substituting the point $(x, y) = (1, 1)$ gives

$$3 - 3 \frac{dy}{dx} = \frac{dy}{dx}$$

Finally solving for $\frac{dy}{dx}$ gives the slope of the desired tangent line: $\frac{dy}{dx} = \frac{3}{4}$.

(b) The revenue from the 6th item is $MR(5)$, which may be approximated by $R'(5)$.

$$R'(x) = \frac{200}{(x + 5)^2} \implies MR(5) \approx R'(5) = \frac{200}{(5 + 5)^2} = 2$$

(c) We use the principle that profit is maximized when marginal cost and marginal revenue are equal. The total revenue is $R(x) = xp(x) = 100x - 3x^2$. Thus we have the following:

$$R'(x) = C'(x) \implies 100 - 6x = 3x^2 + 8x + 60 \implies 3x^2 + 14x - 40 = (3x + 20)(x - 2) = 0$$

Thus profit is maximized when $x = 2$. (We reject the solution $x = -20/3$ since the level of production must be non-negative.

(d) We put $f(x) = x^{1/4}$ and find the linearization (tangent line) of $f(x)$ at $x = 16$. Observe that $f(16) = 2$ and $f'(16) = \frac{1}{4} \cdot 16^{-3/4} = \frac{1}{32}$. So the desired linearization is

$$x^{1/4} \approx 2 + \frac{1}{32}(x - 16)$$
which is a valid approximation if $x$ is close to 16. Hence

$$(16.32)^{1/4} \approx 2 + \frac{1}{32} (16.32 - 16) = 2.01$$

(e) Let $y = x^x$, so that $\ln(y) = x \ln(x)$. Implicitly differentiating this last equation gives

$$\frac{1}{y} \frac{dy}{dx} = 1 + \ln(x)$$

Solving for $\frac{dy}{dx}$ and substituting $y = x^x$ gives

$$f'(x) = \frac{dy}{dx} = x^x (1 + \ln(x))$$

(f) We calculate the limits of $f$ at infinity. For the limit $x \to \infty$, we have the indeterminate form $\frac{\infty}{\infty}$, so we use L'Hôpital’s Rule.

$$\lim_{x \to \infty} \left( \frac{2e^x - 5}{3e^x + 2} \right) = \lim_{x \to \infty} \left( \frac{2e^x}{3e^x} \right) = \frac{2}{3}$$

For the limit $x \to -\infty$ recall that $e^x \to 0$, and so

$$\lim_{x \to -\infty} \left( \frac{2e^x - 5}{3e^x + 2} \right) = \frac{0 - 5}{0 + 2} = \frac{-5}{2}$$

So the two horizontal asymptotes of $f(x)$ are $y = 2/3$ and $y = -5/2$.

---

12 pts 2. Consider the function

$$f(x) = e^{-x^2/2}$$

Find where $f$ is concave down and find where $f$ is concave up. Then find all inflection points ($x$- and $y$-coordinates). Write “NONE” for your answer if appropriate.

Solution

The first two derivatives of $f$ are

$$f'(x) = -xe^{-x^2/2}, \quad f''(x) = (x^2 - 1)e^{-x^2/2}$$

Since $f$ is twice-differentiable on its domain, the only second-order critical numbers are solutions to $f''(x) = 0$.

$$(x^2 - 1)e^{-x^2/2} = 0 \implies x^2 - 1 = 0 \implies x = \pm 1$$

We make a sign chart for $f''(x)$. Note that $e^{-x^2/2}$ is positive for all $x$. So we need only test the sign of $x^2 - 1$. 
Hence $f$ is concave down on the interval $(-1, 1)$ and concave up on the intervals $(-\infty, -1)$ and $(1, \infty)$. There are points of inflection at $(-1, e^{-1/2})$ and $(1, e^{-1/2})$.

**Acceptable answer:** The function $f$ is concave down on $[-1, 1]$ and concave up on $(-\infty, -1]$ and $[1, \infty)$. We may also say that $f$ is concave up on $(-\infty, -1) \cup [1, \infty)$ or $(-\infty, -1) \cup (1, \infty)$.

---

### 3. Consider the function

$$f(x) = \frac{1}{x^2 - 6x}$$

Find all vertical asymptotes of $f$. Then find where $f$ is decreasing and find where $f$ is increasing. Finally determine the $x$-coordinates of all local extrema of $f$ (and classify them as either a local minimum or a local maximum). Write “NONE” for your answer if appropriate.

**Solution**

The domain of $f$ is all real numbers except where $x^2 - 6x = 0$, or $x(x - 6) = 0$. Hence the only numbers not in the domain of $f$ are $x = 0$ and $x = 6$. Since $f$ is algebraic, we know that $f$ is continuous on its domain, hence the only candidates for vertical asymptotes are the lines $x = 0$ and $x = 6$. Direct substitution of either $x = 0$ or $x = 6$ gives the undefined expression “$\frac{1}{0}$”. So we know that all corresponding one-sided limits at $x = 0$ and $x = 6$ must be infinite. So both lines $x = 0$ and $x = 6$ are, indeed, vertical asymptotes.

For intervals of increase and local extrema, we examine the first derivative.

$$f'(x) = \frac{-(2x - 6)}{(x^2 - 6x)^2}$$

Since $f$ is differentiable on its domain, the only first-order critical numbers are solutions to $f'(x) = 0$.

$$\frac{-(2x - 6)}{(x^2 - 6x)^2} = 0 \implies -(2x - 6) = 0 \implies x = 3$$

We make a sign chart for $f'(x)$. Recall that since $x = 0$ and $x = 6$ are not in the domain of $f$, we must include those numbers on our sign chart. (Why? Since there are vertical asymptotes at $x = 0$ and $x = 6$, $f'(x)$ might have different signs to the left and right of each of these $x$-values.)
<table>
<thead>
<tr>
<th>interval</th>
<th>test point</th>
<th>sign of $f'$</th>
<th>shape of $f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>$f'(-1) = \oplus\oplus\ominus$</td>
<td>$\oplus$</td>
<td>increasing</td>
</tr>
<tr>
<td>$(0, 3)$</td>
<td>$f'(1) = \oplus\oplus\ominus$</td>
<td>$\oplus$</td>
<td>increasing</td>
</tr>
<tr>
<td>$(3, 6)$</td>
<td>$f'(5) = \oplus\oplus\ominus$</td>
<td>$\ominus$</td>
<td>decreasing</td>
</tr>
<tr>
<td>$(6, \infty)$</td>
<td>$f'(7) = \oplus\oplus\ominus$</td>
<td>$\ominus$</td>
<td>decreasing</td>
</tr>
</tbody>
</table>

Hence $f$ is decreasing on the intervals $(3, 6)$ and $(6, \infty)$; and $f$ is increasing on the intervals $(-\infty, 0)$ and $(0, 3)$. There is no local minimum, but there is a local maximum at $x = 3$.

*Acceptable answer:* The function $f$ is decreasing on $(3, 6) \cup (6, \infty)$ and $f$ is increasing on $(-\infty, 0) \cup (0, 3)$. (Note that including 0 or 6 in any of these intervals is incorrect.)

**11 pts**

4. The surface area of a sphere is changing at a rate of $16\pi$ in$^2$/s when its radius is 3 in. At what rate is the volume of the sphere changing at that time? *You must include correct units as part of your answer.*

*Hint:* If a sphere has radius $R$, then its surface area $A$ and volume $V$ are given by

$$A = 4\pi R^2, \quad V = \frac{4\pi}{3} R^3$$

**Solution**

Implicitly differentiating each of the given formulas with respect to time $t$ gives

$$\frac{dA}{dt} = 8\pi R \frac{dR}{dt}, \quad \frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}$$

We are given that $\frac{dA}{dt} = 16\pi$ when $R = 3$, and substituting this information into this last pair of equations gives

$$16\pi = 24\pi \frac{dR}{dt}, \quad \frac{dV}{dt} = 36\pi \frac{dR}{dt}$$

Solving for $\frac{dR}{dt}$ in the first of this last pair of equations gives $\frac{dR}{dt} = \frac{2}{3}$, and substituting this value into the second of this last pair of equations gives $\frac{dV}{dt} = 24\pi$. Hence the volume of the sphere is changing at a rate of $24\pi$ in$^3$/s.

**11 pts**

5. A poster is to have a total area of 150 in$^2$, which includes a central printed area, 1-inch margins at the bottom and sides, and a 2-inch margin at the top. What poster dimensions (in inches) will give the largest printed area? Use calculus to justify your answer.
You must demonstrate that your answers really are the optimal dimensions.

Solution
Let $x$ and $y$ be the width and height of the poster, as shown in the diagram. Then our objective is to find the absolute maximum value of the function

$$p(x, y) = (x - 2)(y - 3)$$

which corresponds to the area of the central printed region. The variables $x$ and $y$ are not independent, but rather satisfy the equation (or constraint) $xy = 150$ (total area is 150). Solving for $y$ in the constraint gives $y = 150/x$, and so the area of the central printed region is given by the function

$$f(x) = p\left(x, \frac{150}{x}\right) = (x - 2)\left(\frac{150}{x} - 3\right) = 156 - 3x - \frac{300}{x}$$

Note that the problem requires that $x$ be no smaller than 2 (the minimum width due to the horizontal margins) and $y$ be no smaller than 3 (the minimum height due to the vertical margins). The condition $y \geq 3$ is equivalent to $\frac{150}{x} \geq 3$, or $x \leq 50$. Hence our goal is to find the absolute maximum value of

$$f(x) = 156 - 3x - \frac{300}{x}$$

on the interval $[2, 50]$. Since $f$ is differentiable on this interval, the critical numbers are the solutions to $f'(x) = 0$.

$$0 = f'(x) = -3 + \frac{300}{x^2} \implies x^2 = 100 \implies x = \pm 10$$
(We ignore the critical number $x = -10$ since it is not in $[2, 50]$.) Checking the endpoint values and critical value, we get: $f(2) = 0, f(10) = 96, \text{ and } f(50) = 0$. Hence the area of the printed region has an absolute maximum when $x = 10$ and $y = \frac{150}{10} = 15$.

Alternatively...
Instead of finding the precise interval of allowed $x$-values, we may observe that the allowed interval is some subinterval of $(0, \infty)$ since lengths must be positive. Observe that
\[ f''(x) = \frac{-600}{x^3} \]
and $f''(x) < 0$ for $x > 0$. Hence $f(x)$ is concave down on $(0, \infty)$. So the only critical number $x = 10$ must give rise to a local (and hence absolute) maximum value.
(We may also instead use the first derivative test to determine that $x = 10$ gives a local (and hence absolute) maximum value.)

6. For each part, calculate the limit or show that it does not exist. If the limit is infinite, write “$\infty$” or “$-\infty$” as your answer, as appropriate.

(a) $\lim_{x \to 3^-} \left( \frac{x^2 + 6}{3 - x} \right)$
(b) $\lim_{x \to 0} (1 - \sin(3x))^{1/x}$
(c) $\lim_{x \to -3} \left( (x + 3) \tan\left( \frac{\pi x}{2} \right) \right)$

Solution
(a) Direct substitution of $x = 3$ gives the expression $\frac{15}{0}$, which is not indeterminate, but instead indicates that the one-sided limit is infinite. Observe that the denominator $3 - x$ approaches 0 as $x \to 3^-$, but remains positive. (Recall that the notation $x \to 3^-$ implies $x < 3$.) Hence we have
\[ \lim_{x \to 3^-} \left( \frac{x^2 + 6}{3 - x} \right) = \infty = +\infty \]

(b) Direct substitution of $x = 0$ gives the indeterminate exponent $1^{\pm \infty}$. So we will ultimately use L'Hôpital’s Rule, but we must perform some algebra first to get the expression into the form of a quotient. Put
\[ L = \lim_{x \to 0} (1 - \sin(3x))^{1/x} \]
and consider $\ln(L)$.
\[ \ln(L) = \lim_{x \to 0} \ln \left( (1 - \sin(3x))^{1/x} \right) = \lim_{x \to 0} \left( \frac{\ln (1 - \sin(3x))}{x} \right) \]
Direct substitution of $x = 0$ now gives the indeterminate form $\frac{0}{0}$, and so we may use
L’Hôpital’s Rule.

\[
\ln(L) = \lim_{x \to 0} \left( \frac{1 - \sin(3x) \cdot (-3 \cos(3x))}{1 - 0 \cdot (-3 \cdot 1)} \right) = \frac{1}{1} \cdot (-3 \cdot 1) = -3
\]

(We have used direct substitution in the last step.) We have determined that \(\ln(L) = -3\), whence we find that \(L = e^{-3}\).

(c) Direct substitution of \(x = -3\) gives the indeterminate product \(0 \cdot \infty\). So we will ultimately use L’Hôpital’s Rule, but we must perform some algebra first to get the expression into the form of a quotient.

\[
\lim_{x \to -3} \left( (x + 3) \tan \left( \frac{\pi x}{2} \right) \right) = \lim_{x \to -3} \left( \frac{(x + 3) \sin(\pi x/2)}{\cos(\pi x/2)} \right)
\]

Direct substitution of \(x = -3\) now gives the indeterminate form \(0/0\), and so we may use L’Hôpital’s Rule.

\[
\lim_{x \to -3} \left( \frac{(x + 3) \sin(\pi x/2)}{\cos(\pi x/2)} \right) = \lim_{x \to -3} \left( \frac{(x + 3) \cos(\pi x/2) \cdot (\pi/2) + \sin(\pi x/2)}{-\sin(\pi x/2) \cdot (\pi/2)} \right)
\]

\[
= \frac{0 + (-1)}{-(-1) \cdot (\pi/2)} = \frac{-2}{\pi}
\]

(We have used direct substitution in the last step.)

11 pts 7. A piece of cardboard that is 24 inches wide and 15 inches long is to be used to construct a box with an open top. To do this, congruent squares are cut from each corner of the cardboard, and the flaps are folded up and taped to form the sides of the box. What is the largest possible volume of such a box? Use calculus to justify your answer.

You must demonstrate that your answers really are the optimal dimensions.

\[
\text{Solution}
\]

Let \(x\) be the length of the square that is cut out of each corner. Then the length of the top and bottom flaps is \(24 - 2x\) and the length of the left and right flaps is \(15 - 2x\). Note that the lengths of these flaps are the lengths of the base of the box. Since \(x\) is the width of each flap, \(x\) is also the height of the box. Thus the total volume of the box in terms of \(x\) is

\[
V(x) = (24 - 2x)(10 - 2x)x = 4x^3 - 78x^2 + 360x
\]
Note that the problem requires that the dimensions of the box remain non-negative. That is, we must have that \( x \geq 0, 24 - 2x \geq 0, \) and \( 15 - 2x \geq 0. \) Together, these three inequalities imply that \( x \) must satisfy \( 0 \leq x \leq 7.5. \) (Note that we do allow the degenerate cases of \( x = 0 \) and \( x = 7.5. \) In both of these cases, the resulting “box” has zero volume, but this is mathematically okay and preferable!) Thus our goal is to find the absolute maximum value of \( V(x) \) on the interval \([0, 7.5].\)

Since \( V(x) \) is differentiable for all \( x, \) the only critical numbers satisfy \( V'(x) = 0. \) The derivative \( V'(x) \) satisfies:

\[
V'(x) = 12x^2 - 156x + 360 = 12(x^2 - 13x + 30) = 12(x - 10)(x - 3)
\]

Thus the only critical number of \( V(x) \) is \( x = 3. \) (We must reject \( x = 10 \) since it lies outside the interval \([0, 7.5].\)) Now we use the closed, bounded interval method of Section 4.1 to verify that \( x = 3 \) does, indeed, give the maximum volume. Note that the endpoint values are both 0 (that is, \( V(0) = V(7.5) = 0. \)) Thus, since \( V(3) \) is clearly positive, \( V(3) \) must be the absolute maximum value of \( V(x) \) on \([0, 7.5].\)

Thus the largest possible volume is \( V(3) = 3(24 - 6)(15 - 6) = 3(18)(9) = 486 \text{ in}^3. \)