Chapter 1: the Mathematics of Voting:

1a.
3+5+9+4+2 = **23 people** voted in this election.

1b.
No, there is no majority candidate here, since no candidate has 12 or more first place votes.

1c.
Dave is a Condorcet in this election, since he is ranked higher than Ann on a majority of a ballots, he is ranked higher than Bob on a majority of a ballots, and he is ranked higher than Claire on a majority of a ballots. In other words, he defeats every opponent in head to head competition.

1d.
Bob wins under the plurality method. The ranking is
1st Bob
2nd Ann
3rd Claire
4th Dave

1e.
Ann wins under the Instant Runoff method. The ranking is
1st Ann
2nd Bob (eliminated 3rd)
3rd Claire (eliminated 2nd)
4th Dave (eliminated 1st)

1f.
Dave wins under the Borda count method. The ranking is
1st Dave (68 points)
2nd Claire (59 points)
3rd Bob (54 points)
4th Ann (48 points)

1g.
Dave wins under the pairwise comparisons method. The ranking is
1st Dave (won 3 matches)
2nd Claire (won 2 matches)
3rd Ann (won 1 match)
4th Bob (won 0 matches)

1h.
There is no majority candidate, so none of the results of this particular election violate the majority criterion.
The results of the plurality and Instant Runoff methods do violate the Condorcet criterion, since

Dave is the Condorcet candidate, but is not chosen as winner by those two methods.
Although we say in general that the Borda count method "violates the Condorcet criterion", in the sense that it is vulnerable to the risk of making a Condorcet candidate lose the election, such unfairness happens not to be the outcome of this particular election; here the Borda count method does succeed in selecting the Condorcet candidate as the winner. So the results of *this particular election* under the Borda count method do not violate the Condorcet criterion.
Since the pairwise comparisons method satisfies the Condorcet criterion -- indeed it was designed for this purpose -- it can never cause a Condorcet candidate to lose in any election.

1i. Ann was the winner of the original election, so she is a non-winner (an "irrelevant alternative") who has been disqualified from the revote. The preference schedule for the revote is as follows:

# of voters:	3	5	9	4	2
1st choice	A	C	D	A	D
2nd choice	C	D	C	D	C
3rd choice	D	A	A	C	A

Claire is eliminated first, and then Dave wins. Since the winner has changed from the original election to the revote, due only to the removal of a candidate who did not win the original election, the results of the revote do violate the Independence of Irrelevant Alternatives criterion.
Remark: The disqualification of Bob from the revote is not a step within the Instant Runoff method. On the contrary, it is a piece of external information which is simply given; we find out, for whatever reason that Bob is disqualified. That is not something you are supposed to infer from preference schedule; it is just an extra piece of given information. One you learn that Bob has been disqualified from the revote, you have a whole new election to think about, which does not include Bob as a candidate; applying the rules of the Instant Runoff method to this new election results in declaring Dave the winner.

2. A majority candidate is automatically a Condorcet candidate; by virtue of possessing more than half of the first place votes, the candidate is ranked higher on a majority of ballots than every individual opponent, and this is precisely what it means to win every head to head competition. But a Condorcet candidate need not be a majority candidate, and indeed most of the examples we have seen in this course of Condorcet candidates (including Dave in the previous question on this review sheet) have not been majority candidates.

3a.
There is not enough information to conclude that any fairness criterion has been violated. Ann has a plurality of first place votes (more than any individual opponent), but might not actually have a majority of first place votes, and only if she had a majority of these would the majority criterion demand that she win the election. Based on the given information, Ann might or might not be a Condorcet candidate – we simply don't have enough information to decide.

3b.
Ann certainly is a majority candidate, since she has more than half of all first place votes. Thus her loss of this election violates the majority criterion. Moreover, because she is a majority candidate she is also a Condorcet candidate (see question 1.2 of this review sheet for the reasoning), and therefore her loss of the election also violates the Condorcet criterion.

3c.
The result violates the Independence of Irrelevant Alternatives criterion, since the only change from the original election to the revote is the disqualification of a non-winner of the original election, and the result is a different winner in the revote. The change from the original election to the revote pertains to the number of candidates, not the ranking of candidates on ballots, so the monotonicity criterion does not apply.

3d.
There are no violations here of fairness criteria. Since it is the *winner* of the original election who is disqualified from the revote, the Independence of Irrelevant Alternatives criterion does not apply. Again, the change from the original election to the revote pertains to the number of candidates, not the ranking of candidates on ballots, so the monotonicity criterion does not apply.

3e.
One might think that the monotonicity criterion has been violated, but it has not been. From the original election to the revote there are ballot changes, but they benefit not only Dave (the original winner) but also Ann (a non-winner of the original election). Thus the monotonicity criterion does not apply. If Dave had been the only candidate to move higher on any ballots, and no other changes had occurred from the original election to the revote, then making anyone other than Dave the winner would indeed have violated the monotonicity criterion.
Note that the candidates are the same in the original election and revote, so the Independence of Irrelevant Alternatives criterion does not apply.

3f.
Claire is a Condorcet candidate, since she defeats every opponent in head to head competition. Thus her loss of the election to Dave violates the Condorcet criterion. There is not enough information to determine if she is a majority candidate (which she might or might not be), hence no reason to think that the majority criterion has been violated.

3g.
Since Claire ties Bob in head to head competition, but does not defeat him, she is not a Condorcet candidate. Nor is she a majority candidate; if she had been, she would have been a Condorcet candidate as well, but we know that she is not. Thus there are definitely no violations here of the majority or Condorcet criteria.

4.

The **plurality, Instant Runoff, and pairwise comparisons methods are guaranteed to make Zeke the winner**. This is because he is a majority candidate.

In the case of the plurality method, having a majority of first place votes certainly gives him a plurality of first place votes, so he is declared the winner.

In the case of the Instant Runoff method, having a majority of first place votes guarantees that he will be declared winner right away, at the beginning of the first round; no eliminations even need to be done to find a winner.

In the case of the pairwise-comparisons method, having a majority of first place votes makes him a Condorcet candidate, and therefore the winner of the election. [Make sure you understand why a majority candidate is necessarily also a Condorcet candidate.]

Under the Borda count method, however, it is still possible for a majority candidate to lose. This can be seen from an example, e.g.

	51	49
1st choice	Zeke	Ann
2nd choice	Ann	Bob
3rd choice	Bob	Claire
4th choice	Claire	Dave
5th choice	Dave	Zeke

In this example Ann is the winner under the Borda count method, in spite of Zeke's status as majority and Condorcet candidate.

5.

The voting method must sometimes violate the Monotonicity Criterion.

6.

A majority candidate is necessarily also a Condorcet candidate. The 11 ballots which make C a majority candidate (by giving C more 1st choice votes than all opponents combined), are precisely the 11 ballots which make C a Condorcet candidate (by placing C higher than A on a majority of ballots, namely those 11 ballots, and by placing C higher than B on a majority of ballots, namely those 11 ballots).

7a.

No, there is no majority candidate. We know this, since a majority candidate would also have been a Condorcet candidate, and there is no Condorcet candidate (since Ann and Dave each lose a head to head competition, and Claire and Bob each have a tie, so no one has 3 outright head to head victories).Since there is no majority candidate, there is no chance of violating the Majority Criterion.

7b.
As just explained, there is no Condorcet candidate, so there can be no violation of the Condorcet Criterion.

Chapter 2: the Mathematics of Power

1a.
The winning coalitions are

{ $\underline{P_1}$, P_2, P_3, P_4}
{ $\underline{P_1}$, P_2, P_3}
{ $\underline{P_1}$, P_2, P_4}
{ $\underline{P_1}$, P_3, P_4}
{ $\underline{P_1}$, $\underline{P_2}$}
{ $\underline{P_1}$, $\underline{P_3}$}
{ $\underline{P_1}$, $\underline{P_4}$}

	Banzhaf power index	Shapley-Shubik power index
P_1	7/10 = 70%	18/24 = 75%
P_2	1/10 = 10%	2/24 = 8.33%
P_3	1/10 = 10%	2/24 = 8.33%
P_4	1/10 = 10%	2/24 = 8.33%

The sequential coalitions are:

< P_1, $\underline{P_2}$, P_3, P_4 >	< P_2, $\underline{P_1}$, P_3, P_4 >	< P_3, $\underline{P_1}$, P_2, P_4 >	< P_4, $\underline{P_1}$, P_2, P_3 >
< P_1, $\underline{P_2}$, P_4, P_3 >	< P_2, $\underline{P_1}$, P_4, P_3 >	< P_3, $\underline{P_1}$, P_4, P_2 >	< P_4, $\underline{P_1}$, P_3, P_2 >
< P_1, $\underline{P_3}$, P_2, P_4 >	< P_2, P_3, $\underline{P_1}$, P_4 >	< P_3, P_2, $\underline{P_1}$, P_4 >	< P_4, P_2, $\underline{P_1}$, P_3 >
< P_1, $\underline{P_3}$, P_4, P_2 >	< P_2, P_3, P_4, $\underline{P_1}$ >	< P_3, P_2, P_4, $\underline{P_1}$ >	< P_4, P_2, P_3, $\underline{P_1}$ >
< P_1, $\underline{P_4}$, P_2, P_3 >	< P_2, P_4, $\underline{P_1}$, P_3 >	< P_3, P_4, $\underline{P_1}$, P_2 >	< P_4, P_3, $\underline{P_1}$, P_2 >
< P_1, $\underline{P_4}$, P_3, P_2 >	< P_2, P_4, P_3, $\underline{P_1}$ >	< P_3, P_4, P_2, $\underline{P_1}$ >	< P_4, P_3, P_2, $\underline{P_1}$ >

1b.

- There are no dictators, since there are no one-member winning coalitions.
- P_1 has veto power, and no other player does, since P_1 is the only player who appears in every winning coalition.
- There are no dummies, since every player is critical in at least one winning coalition. Equivalently, every player is pivotal in at least one sequential coalition.

1c.
Banzhaf and Shapley-Shubik do give the same answer, that P_1 has more than its proportional share of power (i.e. more actual power than its share of total weight), while each of the other players has a smaller share of actual power than its share of total weight.

	Banzhaf power index	Shapley-Shubik power index	Share of total weight
P_1	7/10 = 70%	18/24 = 75%	9/18 = 50%
P_2	1/10 = 10%	2/24 = 8.33%	4/18 = 22.22%
P_3	1/10 = 10%	2/24 = 8.33%	3/18 = 16.67%
P_4	1/10 = 10%	2/24 = 8.33%	2/18 = 11.11%

2a.

There are 6 sequential coalitions. Each one contains all 3 players.

$< P_1, \underline{P_2}, P_3>$ $< P_2, \underline{P_1}, P_3>$ $< P_3, \underline{P_1}, P_2>$

$< P_1, \underline{P_3}, P_2>$ $< P_2, P_3, \underline{P_1}>$ $< P_3, P_2, \underline{P_1}>$

	Shapley-Shubik power index
P_1	4/6 = 66.67%
P_2	1/6 = 16.67%
P_3	1/6 = 16.67%

2b.

P_1 has 4 times more power than P_2.

3a.

The given list of winning coalitions in this example happens to be the same as what we saw above in the example [10:9,4,3,2], and therefore the power distributions will be the same, as well as the answer to the question about who has dictator, veto power, or dummy status.

	Banzhaf power index
P_1	7/10 = 70%
P_2	1/10 = 10%
P_3	1/10 = 10%
P_4	1/10 = 10%

3b.

Only P_1 has veto power, and there are no dictators or dummies.

Remark: P_1 has 7 times more power than any other individual player, according to Banzhaf.

4a.
The complete list of winning coalitions is as follows:

{$\underline{P_1}$, $\underline{P_2}$, $\underline{P_3}$, $\underline{P_4}$, P_5, P_6}
{$\underline{P_1}$, $\underline{P_2}$, $\underline{P_3}$, $\underline{P_4}$, P_5}
{$\underline{P_1}$, $\underline{P_2}$, $\underline{P_3}$, $\underline{P_4}$, P_6}
{$\underline{P_1}$, $\underline{P_2}$, $\underline{P_3}$, $\underline{P_4}$}

The veto power players each have 25% of the actual power, while the other two players have none.

4b.
A council member with veto power has infinitely many times more power than a non-veto power member.

5a.
First list all winning coalitions and underline the critical player(s) in each one:
{A, B, C, D}
{$\underline{A}$, B, C}
{$\underline{A}$, B, D}
{$\underline{A}$, C, D}
{$\underline{B}$, $\underline{C}$, $\underline{D}$}
{$\underline{A}$, $\underline{B}$}
{$\underline{A}$, $\underline{C}$}
{$\underline{A}$, $\underline{D}$}

5b.
There are no dictators, since no player has a weight of 11 or greater. There are no veto power players, since no player is critical in the grand coalition. There are no dummies, since each player is critical in at least one winning coalition.

5c.
Districts B and C have a share of power which is slightly less than their share of population; if any districts complain about the unfairness of this weighted voting system, it will be B and C. At the same time, districts A and D have a share of power slightly greater than their share of population; according to Banzhaf's analysis, they benefit from the way in which the actual power is distributed, and do so at the expense of B and C.

	Banzhaf power index (i.e. share of actual power)	Percentage of total population
A	6/12 = 50%	9/20 = 45%
B	2/12 = 16.67%	4/20 = 20%
C	2/12 = 16.67%	4/20 = 20%
D	2/12 = 16.67%	3/20 = 15%

5d.
A quota of 20 would require a unanimous vote in order to pass any motion, giving every player veto power. Note that a quota of 19 or 18 would have the same effect.

6a.
First list all sequential coalitions and underline the pivotal player in each one. We know in advance that with 3 players there will be 3! = 6 sequential coalitions.

<A, B-C, D>
<A, D, B-C>
<B-C, A, D>
<B-C, D, A>
<D, A, B-C>
<D, B-C, A>

	Shapley-Shubik power index (i.e. share of actual power)
A	2/6 = 33.33%
B-C	2/6 = 33.33%
D	2/6 = 33.33%

6b.
As a result of the merger of B and C, district A now has a percentage of actual power (as measured by Shapley-Shubik) which is considerably less than its percentage of total county population. The merged district B-C also has a percentage of actual power which is somewhat less than its percentage of population. District D, however, benefits enormously from the new setup, and has a vastly greater percentage of actual power than its percentage of population.

	Shapley-Shubik power index (i.e. share of actual power)	percentage of total county population
A	2/6 = 33.33%	9/20 = 45%
B-C	2/6 = 33.33%	8/20 = 40%
D	2/6 = 33.33%	3/20 = 15%

7a.
{P, D, F1, F2, S1, S2},
{P, D, F1, F2, S1},
{P, D, F1, F2, S2},
{P, D, F1, F2},
{P, D, F1, S1, S2},
and {P, D, F2, S1, S2}.

	Banzhaf power index
P	6/24
D	6/24
F1	4/24
F2	4/24
S1	2/24
S2	2/24

7b.
The president is 6/2=3 times more powerful than a student, according to Banzhaf.

7c.
The president and dean have veto power. There are no dictators or dummies.

7d.
<P, D, F1, F2, S1, S2> < F2, S1, S2, P, D, F1 > < D, F1, F2, S1, S2, P>

Chapter 3: Mathematics of Apportionment

1a.
seats:__officers_____ states:__precincts_____ population: __incidents_____

1b.
1102/40 = 27.55 incidents per officer.

1c.
The average number of incidents per officer equals 27.55. Equivalently, if every officer dealt with exactly the same number of incidents, and that number didn't have to be a whole number, that number would be 27.55 incidents.

1d.

Precinct	Standard quota
Downtown	16.33
East Side	7.70
North Side	5.48
South Side	10.49

2a.

Precinct	Hamilton apportionment
Downtown	16 officers
East Side	8 officers
North Side	5 officers
South Side	11 officers

2b.

Precinct	incidents per officer
Downtown	450/16 = 28.13
East Side	26.50
North Side	30.20
South Side	26.27

2c.
The North Side has the most incidents per officer. This precinct is the worst served, in the sense that the officers are busier with incidents than officers in other precincts.

3a.
26.4 works as a modified divisor:

Precinct	Jefferson apportionment
Downtown	17 officers
East Side	8 officers
North Side	5 officers
South Side	10 officers

3b.

Precinct	incidents per officer
Downtown	450/17 = 26.47
East Side	26.50
North Side	30.20
South Side	28.90

3c.
It is still the North Side which has the most incidents per officer.

4a.
27.5 works as a modified divisor:

Precinct	Webster apportionment
Downtown	16 officers
East Side	8 officers
North Side	5 officers
South Side	11 officers

4b.

Precinct	incidents per officer
Downtown	450/16 = 28.13
East Side	26.50
North Side	30.20
South Side	26.27

4c.
The South Side has fewest incidents per officer. This is advantageous to the residents of the South Side whose police officers have a smaller caseload on average.

5a.
27.56 works as a modified divisor:

Precinct	Modified quota	Huntington-Hill rounding cutoff	Huntington-Hill apportionment
Downtown	16.328	16.492	16 officers
East Side	7.692	7.483	8 officers
North Side	5.479	5.477	6 officers
South Side	10.486	10.488	10 officers

Important: Note that the North Side has a standard quota which exceeds the Huntington Hill rounding cutoff, so it is rounded up to 6 (even though the ordinary rounding used by Webster would round down). Note also that the South Side is still rounded down, since its modified quota is less than the Huntington Hill rounding cutoff.

6a.

Precinct	Hamilton apportionment of 41 officers
Downtown	17 officers
East Side	8 officers
North Side	5 officers
South Side	11 officers

6b.
This example is not an instance of the Alabama Paradox. When one new seat is added, and no other changes are made, no state loses a seat. Thus there is no paradox here.

7.
There is a Quota Rule violation, since A has an upper quota of 28 seats (this is based on the standard quota), but receives 29 seats in the apportionment, i.e. more than its upper quota.

Chapter 4: Mathematics of Fair Division

1a.
Neil divides 10 Neil-points into the year, i.e. 1/3 of the way into April. The resulting shares are Jan through the first 1/3 of April, and last 2/3 of April through Dec.

1b.
Mary values the share containing the last 2/3 of April through Dec more highly, namely 13 and 1/3 out of 20 Mary-points. More precisely, 2/3 of April is worth 2/3 of 2 Mary-points, i.e. 2 x 2/3 = 4/3 = 1 and 1/3 Mary-points. The rest of the share (May – Dec) is worth 12 Mary points. So the share is worth 13 and 1/3 Mary points.

Computing 13 and 1/3 divided by 20 gives 2/3 of the value of the whole year.
(Computing this on your calculator will give .666666667, i.e. 2/3, but you can work with the decimal if you wish).

1c.
Neil receives a share worth ½ of $300,000, i.e. $150,000.
Mary receives a share worth 2/3 of $240,000, i.e. $160,000.
Thus Mary receives a share worth both a higher fraction of the total value of the year, and a higher dollar value.

2a.

	Value of s_1	Value of s_2	Value of s_3	Value of s_4	Minimum fair share amount in dollars	Bid list
Eve	$150,000	$150,000	$150,000	$150,000	$150,000	s_1, s_2, s_3, s_4
Fred	$200,000	$100,000	$500,000	$100,000	$225,000	s_3
Gail	$200,000	$200,000	$200,000	$200,000	$200,000	s_1, s_2, s_3, s_4
Hank	$150,000	$100,000	$450,000	$200,000	$225,000	s_3

2b.
It could have been either Eve or Gail, and we can't tell which. Each one values each share equally.

2c.
In this method, it doesn't matter who bids highest for a disputed share. The proposed division is unacceptable, since it fails to give Hank a fair share.

2d.
Fred and Hank merge s3 and s4 into a single supershare, and apply the Divider-Chooser method to this supershare.

2e.
Fred values the supershare at $600,000, so he is guaranteed a share worth at least $300,000. This is because the Divider-Chooser method guarantees him at least half the value of the supershare. Since $300,000 is at least equal to his original minimum fair share amount of $225000, it is indeed a fair share for him in relation to the original fair division problem.

2f.
Hank values the supershare at $650,000, so he is guaranteed a share worth at least $325,000. Since $325,000 is at least equal to his original minimum fair share amount of $225000, it is indeed a fair share for him in relation to the original fair division problem.

3a.
Ann divides the year into 3 equally valuable shares, worth 6 Ann-points each. Thus she divides at mid-April and mid-July. The resulting shares are Jan through mid-April, mid-April through mid-July, and mid-July through Dec.

3b.
Only Jan through mid-April is a fair share for Bob (i.e. worth at least 6 Bob-points).
Only mid-July through Dec is a fair share for Chris (i.e. worth at least 4 Chris-points).

3c.
Bob receives Jan through mid-April
Ann receives mid-April through mid-July, and
Chris receives mid-July through Dec.

3d.
Since Bob had only one share on his bid list, and similarly for Chris, there is actually just one way to produce a fair division based on the shares Ann has marked off.
Remark: This is not a standoff!!!

4a.
Bob divides into 5-point shares, namely January through mid-March, mid-March through July, and August through December.

4b.
Only August through December is a fair share for Dave, and only August through December is a fair share for Eve. Thus there is no way to give each player one of the shares marked off by Bob, such that every player receives a fair share.

4c.
The remaining shares are merged into a single supershare, namely mid-March through December. Dave and Eve now apply the Divider Chooser method to the supershare, which is worth 14 points to Dave and 21 points to Eve.

One approach is to make Dave the Divider. Then he divides the supershare into two 7-point shares, namely mid-March through mid-September, and mid-September through December. Eve values these two shares equally; just for the sake of making a specific choice, we can give her mid-September through December.

In the end:

- Bob receives January through mid-March, worth 5/15 = 1/3 of the value of the whole year, i.e. exactly a minimum fair share, as expected.
- Dave receives mid-March through mid-September, worth 7/14 = ½ the value of the whole year. Since this is at least 1/3, it is a fair share.
- Eve receives mid-September through December, worth 10.5 / 24 = 43.75% of the value of the whole year. Since this is at least 1/3, it is a fair share.

5a.
Jason divides at the end of June.

5b.
s_1 (Jan-June) is worth 6 points to Lauren, hence 6/18 = 1/3 of the value of the whole year, while s_2 (July-Dec) is worth 12 points to her, hence 12/18 = 2/3 of the value of the whole year.

5c.
Lauren selects July – Dec., so Jason receives Jan-June at this stage.

5d.
Jason now creates the subshares

j_1: January-February
j_2: March-April
j_3: May- June

each worth 2/12 = 1/6 of the value of the whole year to Jason.

5e.
Lauren now creates the subshares

l_1: July through the first third of October
l_2: last 2/3 of October through the first 2/3 of November
l_3: last 1/3 of November-December

each worth 4/18 = 2/9 of the value of the whole year to Lauren.

5f.
Each of j1, j2, and j3 are worth 2 Kelly points, i.e. 2/9 of the value of the whole year.
l_1: (July through the first third of October) is worth 3 Kelly points, i.e. 3/9 = 1/3 of the value of the whole year.
Both l_2 and l_3: are worthless to Kelly.

5g.
Kelly can select any subshare from Jason, say j_3, and certainly selects l_1 from Lauren.

5h.

Player	**Subshares received**	**Description (which months or parts of months)**	**Fraction of value of the entire year**
Jason	j_1, j_2	January through April	4/12 = 1/3
Kelly	j_3, l_1	May through the first third of October	5/9
Lauren	l_2, l_3	last 2/3 of October through December	8/18 = 4/9

Each of these fractions is at least equal to 1/3, or greater, so every player does indeed receive a fair share.

Chapter 5: Mathematics of Fair Distribution

1a.
For Wendy, $200,000 and for Xavier, $150,000.

1b.
First settlement:

	Items received (if any), and their value	$ initially paid to bank account	$ initially received from bank account
Wendy	House, worth $400,000	$200,000	
Xavier	none		$150,000

1c.
There is a $50,000 surplus, and each player receives $25,000.

1d.
Final settlement:

	Items received (if any), and their value	net $ paid to bank account after the surplus is distributed	net $ received from bank account after the surplus is distributed
Wendy	house, worth $400,000	$200,000 - $25,000 = $175,000	
Xavier	none		$150,000 + $25,000 = $175,000

2a.
First settlement:

	Items received (if any), and their value	$ initially paid to bank account	$ initially received from bank account
Jerry	Couch and coffee maker, worth $2000 total	$2000 - $750 = $1250	
Elaine	none		$600
George	Television, worth $1200	$1200 - $600 = $600	
Kramer	none		$525

2b.
There is a $725 surplus. Each player receives $181.25

2c.
Final settlement:

	Items received (if any), and their value	net $ paid to bank account after the surplus is distributed	net $ received from bank account after the surplus is distributed
Jerry	Couch and coffee maker, worth $2000 total	$1250 - $181.25 = $1,068.75	
Elaine	none		$600 + $181.25 = $781.25
George	Television, worth $1200	$600- $181.25 = $418.75	
Kramer	none		$525 + $181.25 = $706.25

3.
Claire receives her first minimum fair share, LMN.
Bob receives his second minimum fair share, PQ.
Ann receives her third minimum fair share, STU.
Dave receives his fourth minimum fair share, XYZ.

The leftover items are O, R, V, and W. Each one should go to a player, but the method does not specify how this should be carried out.

4.

	A	O	O	P	A	O	A	P	A	P	A	P
Ed	1	1	1	1	1	1	1	1	1	1	1	1
Faith	1	0	0	1	1	0	1	1	1	1	1	1
Greg	1	2	2	1	1	2	1	1	1	1	1	1

G_1 E_1 F_1 G_2 E_2 F_2

A O O | P | A | O A | P | A | P A P

Chapter 6: The Mathematics of Growth and Finance

1a.
$FV = PV(1 + i)^n$ pertains to the growth or decline of a single starting quantity (PV), whether it is a number of dollars or of people or of anything else, into a new quantity FV, based on a succession of percentage changes.

The other two formula pertains to annuities, in which a number of equal payments (each of size *PMT*) are made at regular intervals. It is used when payments are made first, and accumulate in the end to some (usually large) amount.

1b.
In the first formula, *n* is the **number of times growth by a factor of 1+i occurs**.
In the second formula, *n* is the **number of payments made**.

2a.
The overall increase will be by 302.37%.
$1.0475^{30} = 4.023656975 = 1 + 3.023656975$, so the overall % increase will be 302.37% (when rounded to the nearest 100^{th} of a percent).

2b.
The overall decline will be by 76.78%
$(1 - .0475)^{30} = (.9525)^{30} = 0.232246689$. This means that "23.22% of the original population remains after 30 years", or more precisely, that the new population is only 23.22% as large as the original population.
Thus the decline is by 100% - 23.22% = 76.78%.

3a.
There will be $9,621.41 in the account on your 50^{th} birthday.
This is a one-time deposit, which grows by a factor of 1.06, twenty separate times, hence $(\$3000)(1.06)^{20} = \$9621.41$.

3b.
$119, 978.18

Since there are multiple payments first, and then a large accumulated sum at the end, this is a job for

$$FV = PMT\,\frac{(1+i)^n - 1}{i}$$

with i = .06 and n = 21, since there are 21 payments. To appreciate why there are 21 payments

rather than 20, see the sum written out in part c.

3c.
After making 21 payments of $3,000 each, you will have paid a total of

($3,000 per payment)(21 payments) = $63,000

directly into the savings account. This means that the other

$119, 978.18 - $63,000 = **$56, 978.18 will be interest earned.**

3d.
$(\$3,000)(1.06)^{20} + (\$3,000)(1.06)^{19} + (\$3,000)(1.06)^{18} + \ldots (\$3,000)(1.06)^{2} + (\$3,000)(1.06)^{1} + (\$3,000)(1.06)^{0}$
The account value is really the result of summing the future values of many individual transactions, each of which is its own small compound interest problem.

4.
Capital Two has an APY of 34.01%.
Compute $(1 + .3/6)^6 = 1.3401 = 1 + .3401$. There are 6 compounding periods per year.

Lunchers Club Card has an APY of 33.59%.
Compute $(1 + .29/104)^{104} = 1.3359 = 1 + .3359$. There are 104 compounding periods per year, 2 for each week.

Lunchers Club Card offers a better deal for the cardholder, since it has the lower APY.

5.
The price of the house increased by 4% overall.
Compute (1.3)(.8) = 1.04 = 1+ .04. Note that it is important to say explicitly that the 4% change is an *increase*.

6.
You need to save $608.60 each month.

Set up $FV = PMT\frac{(1+i)^n - 1}{i}$

with FV = $50,000 since that is the amount to be accumulated at the end, after a succession of regular payments
i = .06/12 = .005, and
n = 69 payments (9 in 2015 from April through December, then 12 each in the five years from

2016 through 2020)

7a.
You will owe $12,438.95 three years from now.
This is a one time transaction, hence a job for the compound interest formula,
$FV = PV(1 + i)^n$
This is not an annuity; there aren't multiple payments.
Here i = the weekly interest rate =.078/52 = .0014,
n = the number of compounding periods = 156, and
PV = amount of your debt today = $10,000.

FV represents the amount your debt will grow into over the course of 3 years, as a result of being multiplied by 1.0014 every week.

7b.
Your debt will grow by an overall growth factor of $(1.0014)^{156}$ = **1.24389**
This makes sense, since your debt is multiplied by 1.0014 every week, and this multiplication occurs 156 separate times.

8.
Your deposit needs to be $35,618.68.
By contrast to problem 7 above, this involves a one-time deposit (not a succession of regular deposits). Thus it is a job for the compound interest formula, $FV = PV(1 + i)^n$
Here FV = $50,000
i = .06/12 = .005
n = 68, since the money will sit in the bank for 68 full months (8 months from May 1, 2015 to Jan 1, 2015, then another 60 months until Jan 1, 2021).
(Note: n=67 if the year ended on Dec 1, 2020 but since we have 1 month left to earn interest, it is 68.)

9.
APR is the advertised rate of interest for a full year, sometimes called the nominal rate – "nominal" in the sense of "in name only".

By contrast, APY is the rate at which the money in question actually grows over the course of a full year, once the effect of compounding during the year has been taken into account. APY is sometimes called the effective rate, since it is the actual growth rate for the full year.

The distinction is important, because only APY accounts for all relevant information in a single number. One can make an apples to apples comparison between the APYs of competing credit cards, but not between their APRs (which do not reflect all relevant information, namely the effect of compounding during the year).

10a.
The new price after each change can be computed using the corresponding growth factor. This can be done month by month, or in one computation as follows.

$$\$3.60(1 + 0.032)(1 – 0.048)(1 + 0.058)(1 + 0.024) = \$3.83$$

In May the price of gas was $3.83 per gallon.

10b.
The product of the monthly growth factors is 1.064… That is, the 4 month growth factor is 1.064… By subtracting 1 (the original 100%) and converting from decimal to percentage form (multiplying by 100) we find the percent change for the entire period is a 6.439% increase.

11.
Option #1 seems to have the higher return. However, options #2 has more frequent compounding so it may be a better investment. The annual percentage yield (APY) is the necessary information to compare investments with different compounding frequencies.

- Option #1 has an APY of 6.5%
- Option #2 has an APY of $(1 + 0.0645/12)^{12} – 1 = 6.64\%$

Option #2 has the higher return and so is the better investment. I'd rather invest in option #2.

12a.
There is a lot of information in the statement. Let's begin by organizing it.

- $200 is a monthly deposit amount, so it is a PMT
- 4.8% compounded monthly means $i = 0.048/12 = 0.004$
- 36 years from now with monthly deposits means $n = 36(12) = 432$ deposits (or months)

We want to know the value in an account after many regular deposits, the formula for this is $FV = PMT\frac{(1+i)^n - 1}{i}$

Now we can answer the question. There will be $FV = 200\frac{(1+0.004)^{432} - 1}{0.004} = \$230{,}500.69$ in the account.

12b.
The first investment was $200 and earned interest every month except the month it was deposited. That is, it earned interest 431 times. Because we are talking about only one deposit we need $FV = PV(1 + i)^n$. It will be worth $200(1.004)^{431} = \$1{,}117.53$.

13.
Making deposits at the end of each month to reach a goal calls for the formula

$FV = PMT\frac{(1+i)^n - 1}{i}$.
In this case we know FV = \$5,000. We also know i = 0.04/12 and n=5(12) = 60.
Solving for PMT we find $PMT = FV / \frac{(1+i)^n - 1}{i}$ = 5000/ 66.2989… = \$75.42.

Depositing \$75.42 into this account at the end of every month for 5 years will result in a balance of \$5,000.

14a.
Making the deposit at the end of each quarter means that the last deposit will not earn interest. The future value will be only \$1,000.

14b.
Because we want to know the value of an account after many regular deposits, we must use the formula $FV = PMT\frac{(1+i)^n - 1}{i}$.
The PMT is \$1,000.
The interest rate is 4.5% compounded quarterly making i = 0.045/4 = 0.01125.
The deposits are quarterly for 40 years making n = 40(4) = 60.

There is $1{,}000(1.01125^{160} - 1)/0.01125$ = \$443,480.43 in the account when Sally retires.

14c.
The second deposit earned interest less than 160 times. It did not earn interest the first quarter (as it was not yet invested). It did not earn interest the second quarter (as it was invested at the end of that quarter). That leaves 158 times which interest was earned.

We are interested only in a single investment so the formula $FV = PV(1 + i)^n$ is the one for us.
The second investment will contribute $1{,}000(1.01125)^{158}$ = \$5,856.64 to Sally's retirement.

15.
First, we must determine the correct formula to use. We know there are payments, so one of the annuity formulas is called for. We want to know how much Jake can afford to borrow, that is we want to know the present value (PV).

The value for i is 0.033/12 = 0.00275 and for n we have (4 years)(12 months per year) = 48.
Combining this with the payment we have the following.

$$PV = PMT\frac{1}{i}\left(1 - \frac{1}{(1+i)^n}\right)$$

$$PV = 350\,\frac{1}{0.00275}\left(1 - \frac{1}{(1+0.00275)^{48}}\right)$$

PV = \$15,718.21

He can afford to borrow $15,718.21.

16.
This time we want to compute a future value.

$$FV = PMT\frac{(1+i)^n - 1}{i}$$

$$FV = 350\frac{(1+0.029/12)^{48} - 1}{0.029/12}$$

$FV = \$17{,}790.44$

Jake would have $17,790.44.

These last two problems could be car purchasing plans. If Jake saves for 4 years before buying the car instead of taking out a 4 year loan, he'll have an extra $2,000 to spend!

Chapter 7: Graph Theory

1.
The number of distinct Hamiltonian circuits in (18-1)! which is about 3.557×10^{14}. The supercomputer computes 1,000,000 Hamiltonian Circuits per second so it would take the computer 17! / 1,000,000 = 355687428.1 seconds. Then convert from seconds to minutes to hours to days to years.
So 344687428.1 seconds ÷ 60 sec/min ÷ 60 min/hr ÷ 24 hours/day ÷ 365 days/yr = about 11 years.

2a.
There are (5-1)! = 24 Hamiltonian circuits.

2b.
HJLDGH or HGDLJH is the cheapest tour with travel time of 56 minutes.

There are 24 distinct Hamiltonian circuits. However, we can cut the list in half if we count the directions temporarily as non-distinct

Circuit	Total Time	Reverse Direction
H L D G J H	17+11+10+25+5=68	H J G D L H
H L G D J H	17+9+10+22+5=63	H J D G L H
H L J D G H	17+19+22+10+11=79	H G D J L H
H L D J G H	17+11+22+25+11=86	H G J D L H
H L J G D H	17+19+25+10+30=101	H D G J L H
H L G J D H	17+9+25+22+30=103	H D J G L H
H J L G D H	5+19+9+10+30=73	H D G L J H

H J G L D H	5+25+9+11+30=80	H D L G J H
H J D L G H	5+22+11+9+11=58	H G L D J H
H J L D G H	5+19+11+10+11=56	H G D L J H
H D L J G H	30+11+19+25+11=96	H G J L D H
H D J L G H	30+22+19+9+11=91	H G L J D H

2c.
The nearest neighbor tour starting at H is HJLGDH with travel time of 73 minutes.

2d.
Either GLDJHG or JHGLDJ is the cheapest tour with travel time of 58 minutes.

Circuit	Travel Time
H J L G D H	5+19+9+10+30=73
L G D J H L	9+10+22+5+17=63
D G L H J D	10+9+17+5+22=63
G L D J H G	9+11+22+5+11=58
J H G L D J	5+11+9+11+22=58

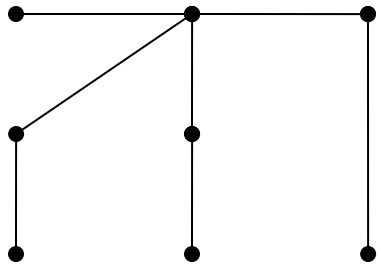
2e.
We can rewrite the starting at H as: HGLDJH or HJDLGH.

2f.
The cheapest link tour starting at H is HJDGLH or HLGDJH with travel time of 63 minutes.

Edge	Weight	Reason
HJ	5	Yes
LG	9	Yes
GD	10	Yes
~~LD~~	~~11~~	No, forms circuit that is not Hamiltonian
JD	11	Yes
LH	17	Yes, Hamiltonian Circuit complete
~~LJ~~	~~19~~	
~~JD~~	~~22~~	
~~JG~~	~~25~~	
~~DH~~	~~30~~	

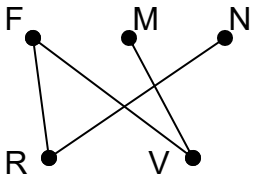
2g.
The Brute-force algorithm is the only algorithm that guarantees the least time tour. (Note: It is not necessarily efficient as we saw in part 2b.)

3a.
There are many possibilities for a spanning tree, below is one. The idea is that you remove enough edges so that there are no circuits remaining in the graph.



3b.
The redundancy is 5. There are 12 edges but with 8 vertices, there should only be 7 edges. So we must remove 12 – 7 = 5 edges.

4a.
The MST of the network is



The work is as follows:

Edge	Weight	Reason
RN	228	Yes
FV	265	Yes
MV	280	Yes
FR	283	Yes – Done Note: (5-1) = 4 edges of spanning tree
~~FM~~	~~304~~	circuit
~~FN~~	~~473~~	circuit
~~RV~~	~~543~~	circuit
~~MR~~	~~584~~	circuit
~~NV~~	~~734~~	circuit
~~MN~~	~~774~~	circuit

4b.
The weight of the MST is 1056 kilometers