RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Jan 2019

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let P be a Sylow p-subgroup of a finite group G and H be a normal subgroup in G.

a) Prove that the intersection of P and H is a Sylow p-subgroup in H.

b) Find an example showing that for non-normal subgroups H the statement a) may not be valid.

2. A ring is called completely left reducible if it is a direct sum of left ideals which are simple modules over the ring. For what integers n is the ring $\mathbb{Z}/n\mathbb{Z}$ completely left reducible?

3. Let A and B be operators in complex finite-dimensional vector space such that $AB - BA = B$.

a) Prove that for all integer $k > 0$ there holds $AB^k - B^kA = kB^k$.

b) Prove that operator B is nilpotent.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Show that the groups of automorphisms of the finite abelian groups $\mathbb{Z}/30\mathbb{Z}$ and $\mathbb{Z}/15\mathbb{Z}$ are isomorphic.

5. Let R be an associative ring with identity. Assume that R has no proper one-sided ideals. Prove that R is a skew-field.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Part I. Answer all questions.

1. Prove that the initial value problem

$$zf''(z) + f'(z) + zf(z) = 0, \quad f(0) = 1.$$

has a unique solution in the form of a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Determine the radius of convergence of this power series.

2. Let f be a function holomorphic in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and continuous in the closure $\overline{\mathbb{D}}$. Prove that

$$\int_{-1}^{0} f(x)dx = -\frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(z) \log(z) dz$$

where $\log(z)$ is the branch of the logarithm function which is holomorphic in $\mathbb{C} \setminus \{(-\infty, 0]\}$ with $\log(1) = 0$.

3. Let f be an entire function in the complex plane such that $|f'(z)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Prove that there exist complex numbers a and c with $|c| \leq 1$ such that $f(z) = ae^{cz}$ for all $z \in \mathbb{C}$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $\mathbb{H}$ stand for the upper half plane in the complex plane $\mathbb{C}$. Suppose that $f : \mathbb{H} \mapsto \mathbb{H}$ is holomorphic. Prove that for any $z \in \mathbb{H}$, $\frac{|f'(z)|}{\mathfrak{Im} f(z)} \leq \frac{1}{\mathfrak{Im}(z)}$, and that if equality holds at some $z_0 \in \mathbb{H}$, then $f(z) = \frac{az+b}{cz+d}$ for all $z \in \mathbb{H}$, for some $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$.

5. Let $(P(x, y), Q(x, y))$ be a C^1 vectorfield in $\mathbb{R}^2 \setminus \{(0, 0)\}$. It is said to be curl free in $\mathbb{R}^2 \setminus \{(0, 0)\}$ if $\partial_x Q(x, y) - \partial_y P(x, y) = 0$ there; it is said to have a potential function in $\mathbb{R}^2 \setminus \{(0, 0)\}$ if there exists a C^2 function $\phi(x, y)$ in that region such that $(P(x, y), Q(x, y)) = (\partial_x \phi(x, y), \partial_y \phi(x, y))$ there.

(a) Prove that $(P(x, y), Q(x, y))$ is curl free in $\mathbb{R}^2 \setminus \{(0, 0)\}$ iff for any $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $\int_{\partial D_r(x_0, y_0)} P(x, y)dx + Q(x, y)dy = 0$ for all $0 < r < \sqrt{x_0^2 + y_0^2}$, where $D_r(x_0, y_0)$ is the disc of radius r centered at (x_0, y_0).

(b) Prove that for any curl free vectorfield $(P(x,y), Q(x,y))$ in $\mathbb{R}^2 \setminus \{(0,0)\}$, there exists a unique $c \in \mathbb{R}$ such that

$$(P(x,y), Q(x,y)) - c(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2})$$

has a potential function in $\mathbb{R}^2 \setminus \{(0,0)\}$.

End of Session 2

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Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let $f \in L^1(\mathbb{R})$. We define the function g by

$$g(\alpha) = \int_{-\infty}^{\infty} f(x-\alpha)\frac{dx}{1+x^2}.$$

Prove that for all such f the function g is continuous. You must state carefully (but without proofs) all theorems you use.

2. Let

$$\tilde{\chi}_{[-1,1]}(x) = \begin{cases} 1, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

and let I be a measurable subset of $\mathbb{R}$. We define

$$I(x) = \int_I \frac{\tilde{\chi}(x-y)}{1+y^2}\, dy.$$

(a) Prove that $I(x)$ is a nonnegative bounded L^1 function on $\mathbb{R}$.

(b) For $n \geq 1$ we define

$$a_n = \int_{-\infty}^{\infty} \cos(n^2 x) I(x)\, dx.$$

Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

3. (a) Let S be a **compact** metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements in S which has a finite number of accumulation points. We denote these points by $y_1, \ldots, y_k$. Prove that it is possible to split the index set $\mathbb{N}$ into a disjoint union of k sets $S_j, j = 1, \ldots, k$ such that each sub-sequence $\{x_n, n \in S_j\}$ converges to y_j.

(b) Let S be **any** metric space and the sequence $\{x_n\}$ has infinite but countable number of accumulation points $y_1, y_2, \ldots$. Prove that it is possible to split the index set $\mathbb{N}$ into a disjoint union of sets $S_j, j = 1, 2, \ldots$ such that each sub-sequence $\{x_n, n \in S_j\}$ converges to y_j.

Part II. Answer one of the two questions.

If you work on both questions, indicate clearly which one should be graded.

4. (a) Suppose that $a(x)$ is a bounded measurable function on $[0,1]$, and $u(x)$ is an absolutely continuous function on $[0,1]$, which satisfies $u'(x) = a(x)u(x)$ for *a.e.* $x \in [0,1]$. Further suppose that $u(0) = 0$. Prove that $u(x) \equiv 0$ on $[0,1]$.

(b) Provide an example to show that the statement of part (a) does not hold if the absolute continuity condition on $u(x)$ is weakened. Namely, exhibit a bounded measurable $a(x)$ on $[0,1]$, $u(x)$ continuous on $[0,1]$ with $u'(x)$ existing and satisfying $u'(x) = a(x)u(x)$ for *a.e.* $x \in [0,1]$, but $u(x) \not\equiv 0$ on $[0,1]$.

5. Assume that $\{f_n\}$ is a sequence of elements of $L^2(0,1)$ which satisfy

$$\sup_n ||f_n||_{L^2(0,1)} < \infty.$$

Further assume that there exists a function $f : (0,1) \to \mathbb{R}$ with $f_n \to f$ a.e. in $(0,1)$.

(a) Prove that $f \in L^2(0,1)$.

(b) Prove that for any $g \in L^2(0,1)$ there holds

$$\lim_{n\to+\infty} \int_0^1 f_n g \, dx = \int_0^1 f g \, dx.$$

End of Session 3