Rutgers University - Graduate Program in Mathematics

Written Qualifying Examination

Fall 1999

Day 1

This exam will be given in two three-hour sessions, one today and one tomorrow. At each session the exam will have two parts. Answer all three of the questions on Part I and three of the six questions on Part II. If you work on more than three questions on Part II, indicate clearly which three should be graded.

First Day—Part I: Answer each of the following three questions.

1. Let $p(x)$ and $q(x)$ be relatively prime polynomials with real coefficients, of strictly positive degrees d_1 and d_2, respectively. Let $r(x)$ be a polynomial of degree less than $d_1 + d_2$. Show that the rational function $\dfrac{r(x)}{p(x)q(x)}$ has a *partial fraction* decomposition of the form

$$\frac{a(x)}{p(x)} + \frac{b(x)}{q(x)} ,$$

where $a(x)$ and $b(x)$ are polynomials of degrees less than d_1 and d_2, respectively.

2. A function $f : \mathbb{C} \to \mathbb{C}$ is said to satisfy a *(global) Lipschitz condition* if there exists a constant K for which $|f(z) - f(w)| \leq K|z - w|$ for all $z, w \in \mathbb{C}$. Show that an entire (i.e., holomorphic) function that satisfies a global Lipschitz condition must have the form $f(z) = az + b$ for some complex constants a and b.

3. Let $U \subset \mathbb{R}$ be a (possibly infinite) open interval and $f : U \to \mathbb{R}$ a differentiable function whose derivative is nondecreasing: $\xi_1 < \xi_2 \Rightarrow f'(\xi_1) \leq f'(\xi_2)$. Prove carefully that for every $[a, b] \subset U$,

(a) $\max\{ f(x) \mid a \leq x \leq b \} = \max\{ f(a), f(b) \}$;

(b) If $\ell(x)$ is the linear function such that $\ell(a) = f(a)$ and $\ell(b) = f(b)$, then $f(x) \leq \ell(x)$ holds for all $a \leq x \leq b$.

First Day—Part II: Answer <u>three</u> of the following six questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Suppose that the finite group G acts on the finite set Ω. Let $n = |G|$ and $m = |\Omega|$. Assume that G has exactly r orbits on Ω, and that their cardinalities are $m_1, \ldots, m_r$. An element $\alpha \in \Omega$ is chosen randomly, and independently of this an element $g \in G$ is chosen randomly, with all choices equally likely in each case. Determine, with a proof, the probability that $g\alpha = \alpha$.

5. Suppose that g is a bounded measurable function defined on $\mathbb{R}$, and that for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, $f(z)$ is defined by

$$f(z) = \int_{-\infty}^{\infty} \frac{g(y)}{1 + zy^2}\, dy.$$

Prove that f is holomorphic in the right half plane $\{\, z \in \mathbb{C} \mid \operatorname{Re} z > 0 \,\}$.

6. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ for $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

(a) Prove that f is continuous on $\mathbb{R}^2$ and that $(D_1 f)(x, y) \equiv \frac{\partial f}{\partial x}(x, y)$ and $(D_2 f)(x, y) \equiv \frac{\partial f}{\partial y}(x, y)$ exist for all $(x, y) \in \mathbb{R}^2$ and are bounded on $\mathbb{R}^2$.

(b) Prove that for every unit vector $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, the directional derivative $(D_{\mathbf{u}} f)(0, 0) \equiv \lim\limits_{t \to 0} \dfrac{f(t\mathbf{u}) - f(0, 0)}{t}$ exists.

(c) Prove that f is *not* differentiable at $(0, 0)$.

7. Define the *exponential* e^A of a square matrix A with complex entries by the standard power series:

$$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots\ .$$

Show:

(a) For any such matrix A, this infinite series converges.

(b) If A and B are similar matrices, then e^A and e^B are similar matrices.

(c) $\det(e^A) = e^{\operatorname{tr}(A)}$. (Here *tr* denotes the trace.)

8. Suppose that $f(z)$ is holomorphic in the unit disc $D = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$, that $f(0) = 0$, and that $\operatorname{Re}[f(z)] < 1$ for $z \in D$. Show that

$$|f(z)| \leq \frac{2|z|}{1 - |z|}$$

for all $z \in D$.

9. Evaluate the following limits rigorously, justifying your answers with the aid of appropriate theorems.

(a)

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)\, dx.$$

(Hint: Prove that $\left(1+\frac{x}{n}\right)^{-n} \leq \left(1+x+\frac{x^2}{4}\right)^{-1}$ for $x \geq 0$ and $n \geq 2$.)

(b)

$$\lim_{n\to\infty} \int_0^n \left(1-\frac{x}{n}\right)^{n} e^{x/2}\, dx.$$

(Hint: Prove that $\left(1-\frac{x}{n}\right)^{n} \leq e^{-x}$ for $n \geq 1$ and $0 \leq x \leq n$.)

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Day 2

This is the second of two three-hour exam sessions; you should have taken the first session yesterday. The exam has two parts. Answer all three of the questions on Part I and three of the six questions on Part II. If you work on more than three questions on Part II, indicate clearly which three should be graded.

Second Day—Part I: Answer each of the following three questions.

1. This problem concerns Lebesgue integration of functions from $\mathbb{R}$ to $\mathbb{R}$. For such functions, a trivial form of Hölder's inequality says that if $f \in L^\infty(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ then

$$\left| \int_{\mathbb{R}} fg \right| \le \|f\|_\infty \|g\|_1. \tag{$*$}$$

(a) Prove that for each $g \in L^1(\mathbb{R})$ there is an $f \in L^\infty(\mathbb{R})$ with $\|f\|_\infty = 1$ such that equality holds in $(*)$.

(b) Prove that for each $f \in L^\infty(\mathbb{R})$ and each $\epsilon > 0$ there is a $g \in L^1(\mathbb{R})$ with $\|g\|_1 = 1$ such that

$$\left| \int_{\mathbb{R}} fg \right| \ge \|f\|_\infty - \epsilon.$$

2. Let $P(z)$ be a polynomial in z of degree two or higher. Let $\mathcal{C}_R$ be the positively oriented semicircle of radius R, with center at the origin, in the upper half plane. Show that

$$\lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{e^{ikz}}{P(z)}\, dz = \begin{cases} 0, & \text{if } k \ge 0, \\ 2\pi i \displaystyle\sum_{n=1}^{m} \operatorname{Res}_{\zeta_n} \frac{e^{ikz}}{P(z)}, & \text{if } k < 0, \end{cases}$$

where $\zeta_1, \ldots, \zeta_m$ are the distinct zeros of $P(z)$ and $\operatorname{Res}_\zeta f(z)$ is the residue of $f(z)$ at ζ.

3. Let G be a group and let H be the subgroup of G generated by the set $\{\, x^2 \mid x \in G \,\}$ of all squares in G.

(a) Show that H is a normal subgroup of G.

(b) Show that G/H is abelian.

(c) Let a, b, c, and d be elements of G. Using (b), deduce that

$$abcdcccbadacdcad$$

is an element of H.

Second Day—Part II: Answer <u>three</u> of the following six questions. If you work on more than three questions, indicate clearly which three should be graded.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is called 1-periodic if $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Let V be the vector space of all 1-periodic C^∞ functions $f : \mathbb{R} \to \mathbb{R}$, let $h : \mathbb{R} \to \mathbb{R}$ be a fixed 1-periodic C^∞ function, and consider the linear differential operator $T : V \to V$ defined by

$$(Tf)(x) = f''(x) + h(x)f(x).$$

Show that any finite-dimensional T-invariant subspace of V is the span of certain one-dimensional T-invariant subspaces. (Hint: use the inner product $(f,g) = \int_0^1 f(x)g(x)\,dx$.) Recall that a subspace W of V is T-invariant if and only if $T(W) \subseteq W$.

5. Suppose that $a_k = \displaystyle\int_0^1 \frac{\cos k\pi x}{\sqrt{x}}\,dx$ for $k = 0, 1, 2 \ldots$. Show that $\displaystyle\sum_{k=0}^{\infty} a_k^2 = \infty$.

6. Let $P(z)$ be a polynomial, with $P(0) \neq 0$, whose complex zeros, enumerated according to multiplicity, are $\{a_k\}_{k=1}^n$. Find expressions for the sums $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k}$ and $\displaystyle\sum_{k=1}^{n} \frac{1}{a_k^2}$ in terms of $P(0)$, $P'(0)$, and $P''(0)$. (Hint: Begin by finding an expression for $\displaystyle\sum_{k=1}^{n} \frac{1}{z - a_k}$.)

7. A metric space is called *separable* if it contains a countable dense set. Let A be a subset of a separable metric space (X, d). Prove that A (with the metric inherited from X) is separable.

8. This problem concerns polynomials and rational functions in two variables, t and u, over the complex field $\mathbb{C}$. Suppose that

$$f(t,u) = \sum_{i=0}^{n} \alpha_i(u)t^i \qquad \text{and} \qquad g(t,u) = \sum_{j=0}^{m} \beta_j(u)t^j\,,$$

where all α_i and β_j are rational functions of u ($\alpha_i, \beta_j \in \mathbb{C}(u)$) and $\alpha_n(u) = \beta_m(u) = 1$. Show that

$$fg \in \mathbb{C}[t,u] \iff f \in \mathbb{C}[t,u] \text{ and } g \in \mathbb{C}[t,u].$$

9. Let $\theta : [0,1] \to [0,1]$ be a strictly increasing, continuously differentiable function with $\theta(0) = 0$ and $\theta(1) = 1$. In this problem we write $V[F; a, b]$ for the total variation of the function F over the interval $[a, b]$.

(a) Let $f : [0,1] \to [0,1]$ be a function of bounded variation and let $g = \theta \circ f \circ \theta^{-1}$. Show, directly from the definition of total variation, that there exists a constant K depending only on θ such that

$$V[g; 0, 1] \le K \; V[f; 0, 1].$$

(b) Determine, with a proof, the constant K in (a) which is optimal in the sense that for any any $K' < K$ there exists an f with $V[g; 0, 1] > K' \; V[f; 0, 1]$.