RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination

Spring 2001, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Prove that if λ is an eigenvalue of an orthogonal matrix then λ^{-1} is also an eigenvalue of the same matrix.

2. Let $f \in L^1([0,1])$. Let $E_1, E_2, E_3, \ldots$ be a sequence of measurable subsets of $[0,1]$ such that $\cap_{n=1}^{\infty}\big(\cup_{k=n}^{\infty} E_k\big) = \emptyset$. Prove:

$$\lim_{n\to\infty} \int_{E_n} f(x)\, dx = 0.$$

3. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)^2} dx.$$

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Construct a conformal mapping from the interior of the unit disc to the interior of the intersection of two disks of radius $\sqrt{2}$ centered at -1 and 1.

5. Suppose that
$$P(z) = z^n + b_1 z^{n-1} + ... + b_n$$
is a polynomial with integer coefficients having all of its roots on the unit circle in the complex plane. Prove that any root of $P(z) = 0$ is a root of unity.

6. Let $f \in L^2(\mathbf{R})$. Prove:
$$\lim_{x \to \infty} \int_x^{x+1} f(t)\, dt = 0.$$

7. Suppose $f_n(z) = \sum_{i=0}^{\infty} a_{i,n} z^i$ are functions so that $|a_{i,n}| \leq Ki$ for all i, n. Show that
   a) for each n, $f_n(z)$ is analytic in the open unit disc,
   b) $\{f_n\}$ forms a normal family in the unit disc.

8. **Construct** a dense G_δ-subset of $(0, 1)$ that has (one-dimensional) Lebesgue measure zero.

9. Let A be the Abelian group of quintuples of integers under addition and let G be the subgroup generated by the elements
$$(1, -1, 0, 2, 1),\ (2, 3, 1, 1, 0),\ (4, 1, 0, 0, 2),\ (-1, 1, -1, 1, 1),\ (1, 1, 1, 1, 1).$$
Determine the group A/G as a product of cyclic groups.

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Spring 2001, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. How many zeros does the polynomial $P(z) = 9z^4 - 2z^3 + 2z^2 - 2z + 2$ have inside the unit circle?

2. Let $U \subseteq \mathbb{R}^2$ be an open connected set. A bounded nonnegative Lebesgue-measurable function $f : U \to \mathbb{R}^+$ is said to be **submedian on squares** if for every square $Q = [a - \epsilon, a + \epsilon] \times [b - \epsilon, b + \epsilon] \subseteq U$, the relation

$$f(a, b) \leq \frac{1}{4\epsilon^2} \int_Q f dA$$

holds, where dA is 2-dimensional Lebesgue measure: in words, the value of f at the center of each closed square contained in U is smaller than or equal to the average of f over the square.

   a) **Prove** that if f is submedian on squares, then f^p is submedian on squares, for $1 \leq p < \infty$.

   b) **Prove** that if a continuous function f is submedian on squares, then f cannot take an interior maximum in U unless f is constant; that is: if there is an $(x_0, y_0) \in U$ for which $f(x_0, y_0) \geq f(x, y)$ holds for every $(x, y) \in U$, then f is identically constant in U.

3. Let α, β be complex numbers such that

$$\alpha^2 = 2\beta - \alpha, \ \alpha\beta = \alpha - 4, \ \beta^2 = \beta - 2\alpha - 2.$$

Show that the set $R = \{a + b\alpha + c\beta \mid a, b, c \text{ integers}\}$ with the usual addition and multiplication of complex numbers is a ring and find all ring homomorphisms from R to the ring with two elements. Determine all ideals I in R for which the quotient ring R/I has two elements.

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $f \in L^1([0,1])$. Let $\varepsilon > 0$. Prove the existence of a compact set K and a polynomial p such that the Lebesgue measure of K exceeds $1-\varepsilon$ and

$$|f(x) - p(x)| < \varepsilon \quad \text{for all} \quad x \in K.$$

5. Show that the function $f(z)$ given by

$$f(z) = \sum_{k=0}^{\infty} z^{2^k}$$

is analytic in the interior of the unit disk $|z| < 1$, but cannot be analytically continued at any point on the boundary of the unit disk.

6. Suppose that ψ is a solution to Liouville's equation

$$\frac{\partial^2 \psi}{\partial u^2} + \frac{\partial^2 \psi}{\partial v^2} = e^{\psi}$$

in a domain D of the (x, y)-plane, and f is a conformal mapping of D to a domain $\tilde{D}$ of the (u, v)-plane. Show that there is a solution φ of

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = e^{\phi}$$

in $\tilde{D}$ which can be written as $\varphi = \psi \circ f + H$ for some function H which is harmonic in D.

7. Let G be a finite group. Find a formula in terms of the order of G and the number of conjucacy classes in G for the probability that two independent random elements of G commute. What is this probability for the group of permutations of five objects?

8. Let $Q_n(x)$ be the degree n polynomial

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

How many real roots does the equation $Q_n(x) = 0$ have?

9. Prove that if $\{a_n\}_{n=0}^{\infty}$ is a sequence for which $\lim_{n\to\infty} a_n = a$, then

$$\lim_{t\to+\infty} e^{-t} \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = a$$

as $t \to +\infty$ through real values. {Hint: use the relations $\sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} \equiv 1$, and for any natural number N, $\lim_{t\to+\infty} \sum_{n=0}^{N} e^{-t} \frac{t^n}{n!} = 0$ and (therefore) $\lim_{t\to+\infty} \sum_{n>N} e^{-t} \frac{t^n}{n!} = 1$.}