RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2006, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Prove that there is no integrable function $\delta : [0,1] \to \mathbb{R}$ with the property that $\int_0^1 \delta(x)f(x)dx = f(0)$ holds for all continuous functions $f : [0,1] \to \mathbb{R}$.

2. Calculate by the Residue theorem the following integral:

$$\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$$

Hint: Consider $f(z) = \dfrac{(\log z)^2}{(1+z^2)^2}$.

3. Can the following matrix A be brought into Jordan canonical form over the field of rational numbers?

$$A = \begin{pmatrix} -3 & -1 & -1 \\ 6 & 4 & 1 \\ 6 & 5 & 0 \end{pmatrix}.$$

If your answer is yes, find a matrix B in Jordan canonical form such that B is similar to A. Be sure to justify your work.

Exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Assume that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ and use it to prove that the sequence

$$F_n(x) = \int_{-\infty}^{x} \frac{\sin nt}{\pi t} dt$$

converges pointwise as $n \to \infty$ to

$$H(x) = \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$$

5. Let $f(z)$ be a holomorphic function in $D = \{z : |z| > 1\}$. Assume that $|f(z)| \leq C(1 + |z|^\alpha)$ for constants $C > 0$ and $\alpha < 1$. Show that ∞ is a removable singular point for $f(z)$.

6. Prove that the ring $\mathbb{Q}[x]/(x^7 - 1)$ is a direct product of two fields. (Terminology: The direct product of rings is also called the product ring.)

7. Suppose X is a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \backslash X$ is path connected.

8. Let $G = SL(2, \mathbb{R})$, the group of 2 by 2 real matrices of determinant 1.

1. Prove that every matrix g in G can be uniquely written in the form $g = bk$, where b is an element of the group B of 2×2 upper triangular real matrices with positive diagonal entries, and k is an element of the group $SO(2, \mathbb{R})$ of 2×2 orthogonal real matrices of determinant 1. Hint: Use Gram-Schmidt.

2. Show that B and $SO(2, \mathbb{R})$ are both path-connected as subspaces of G. Deduce that G is connected.
   (The topology on G is induced from the metric topology on the 4-dimensional vector space of all 2 by 2 real matrices.)

Exam continues on next page

9. An entire function $f(z)$ has said to have *finite order* if there exist positive constants c and n such that

$$|f(z)| < c\ e^{(|z|^n)}.$$

Prove that if such a function has only a finite number of zeroes, then it must be of the form

$$f(z) = p(z)\ e^{q(z)},$$

where p and q are polynomials.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2006, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Is it possible to have a sequence $\{a_{i,j}\}$ of real numbers indexed by natural numbers i and j such that

$$\sum_{j=1}^{\infty} a_{i,j} = b_i$$

exists for every i and $s = \sum_{i=1}^{\infty} b_i$ exists as well, and similarly

$$\sum_{i=1}^{\infty} a_{i,j} = c_j$$

exists for every j and $\hat{s} = \sum_{j=1}^{\infty} c_j$ exists, but such that $s \neq \hat{s}$? Justify your answer. If it is in fact possible, construct an example.

2. Find the Laurent expansion of $f(z) = \dfrac{2}{z^2 + 5z - 6} + e^z$, centered at $z = 1$.

3. Prove that there is no simple group of order 148.

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Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let m be the Lebesgue measure on the real line $\mathbb{R}$ and let $\mathcal{G}$ be a countable family of Lebesgue measurable functions defined on $E \subset \mathbb{R}$, with $m(E) < \infty$. Assume that for every $x \in E$ there exists a finite $M_x > 0$ such that $|f(x)| \leq M_x$ for all $f \in \mathcal{G}$. Show that for any $\epsilon > 0$ there is a closed set $F \subset E$ and a finite $M > 0$ such that $m(E \setminus F) < \epsilon$ and $|f(x)| \leq M$ for all $f \in \mathcal{G}$ and all $x \in F$.

5. Find a conformal mapping that maps the complement of the circular arc $|x + iy| = 1, x \geq 0, y \geq 0$ onto the interior of the unit disk.

6. Let n be a positive integer. Classify the finitely generated modules over the ring $\mathbb{Z}/n\mathbb{Z}$. (Don't forget the case $n = 1$.) Justify your answer.

7. Let $\mathbb{Q}$ denote the rational numbers, viewed as a subspace of the real numbers $\mathbb{R}$. Construct a homeomorphism $f : \mathbb{Q} \to \mathbb{Q}$ that is *not* a strictly monotone function.

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8. Consider the subset T of the set of 5 by 5 real matrices consisting of the matrices of the following form:

$$\begin{pmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{pmatrix}.$$

Let A be an invertible matrix in the subset T. Show that the inverse of A is also in T.
Hint: Express A as $aI + bZ + cZ^2 + dZ^3 + eZ^4$, where Z is a permutation matrix of order 5. Do not attempt to give a detailed formula for the inverse of A.

9. Let X be a compact topological space, and J a family of continuous real-valued functions on X which is closed under addition and multiplication. Show that either J contains a nowhere-vanishing function or that there is a point x in X at which every function in J vanishes.

Exam Day 2 End