RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2007, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Let F be a field. Prove that for any element M in the ring of n by n matrices over F exactly one of the following holds: (1) M is invertible, (2) M has a left and right zero divisor. (You may assume standard results from elementary linear algebra.)

2. The function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

has various Taylor and Laurent series which converge in disks or annuli centered in the origin. Find all of these series and for each give the domain on which it represents the function.

3. Let $\{f_k\}$ and $\{g_k\}$ be sequences of measurable functions which are finite (no points are mapped to ∞ or $-\infty$) and converge in measure to finite measurable functions f and g, respectively, on a measurable set E with finite measure. Show that $\{f_k g_k\}$ converges in measure to fg.

The exam continues on next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let (X, d) be a compact metric space. Let $f : X \to X$ be a map such that $d(f(x), f(y)) = d(x, y)$. Prove that

(a) f is a continuous bijection

(b) f is a homeomorphism.

5. A subgroup H is maximal in a group G if $H \neq G$ and every subgroup of G containing H coincides with H or G. Prove that the intersection of two different commutative maximal subgroups is contained in the center of the group.

6. Let D be the portion of the right half plane $\mathrm{Re}\, z \geq 0$ lying outside the circle $|z - 2| = 1$. (a) Find a conformal mapping of D onto an annulus $r < |z| < 1$. (b) Find a continuous bounded function on $\bar{D}$ which is harmonic in D, vanishes on the imaginary axis, and takes value 1 on $|z - 2| = 1$.

7. Use the Residue Theorem to evaluate

$$\int_0^\pi \frac{\cos t \, dt}{1 - 2a \cos t + a^2},$$

where $-1 < a < 1$.

8. Prove that

$$\frac{d}{dx} \int_0^\infty f(y) \sin(xy) \, dy = \int_0^\infty y f(y) \cos(xy) \, dy$$

if $(1 + y) f(y) \in L(0, \infty)$.

The exam continues on next page

9. Prove that the rings $\mathbb{Z}_{mn}$ and $\mathbb{Z}_m \oplus \mathbb{Z}_n$ are isomorphic if and only if m and n are mutually prime, i.e. $\gcd(m, n) = 1$.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2007, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let X and Y be connected topological spaces. Let A and B be proper subsets of X and Y respectively. Endow $X \times Y$ with the product topology. Prove that

$$S = (X \times Y) \setminus (A \times B)$$

is connected in the induced topology.

2. Prove that the center of a group of order p^m (p is prime, $m > 1$) is not trivial.

3. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions defined and continuous in the closed upper half plane, analytic in the open upper half plane, and such that $\lim_{|z|\to\infty} f_n(z)$ exists. Suppose that $|f_n(x)| \leq a_n$ for x real, where $\sum a_n$ converges. Prove that $\sum f_n$ converges to a function analytic in the open upper half plane.

The exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Consider $\mathbf{R}^{\mathbf{N}}$, the countable Cartesian product of the real numbers. Endow $\mathbf{R}^{\mathbf{N}}$ with the box topology, that is, the basis open sets W in this topology are of the form

$$W = \Pi_{i=1}^{\infty}(a_i, b_i).$$

Consider the set S,

$$S = \{(a_1, a_2, a_3, \dots),\ a_i > 0\}.$$

Prove that $x_0 = (0, 0, 0, \dots)$ is a limit point of S, but that no sequence from S converges to x_0.

5. Prove that any matrix over $\mathbf{C}$ is similar to its transpose matrix.

6. Let R be a commutative ring with identity and without zero divisors, i.e. R is an integral domain. Consider R as a module over itself. Prove that R is isomorphic to every one of its nonzero submodules if and only if R is a principal ideal domain (PID).

7. Suppose that Ω is a connected open subset of $\mathbb{R}^2 = \mathbb{C}$ and that $u, v : \Omega \to \mathbb{R}$ are continuously differentiable and satisfy $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ throughout Ω. Prove that the function $f(x+iy) = u(x,y) + iv(x,y)$ is analytic in Ω.

The exam continues on next page

8. (a) Show that an analytic function $f(z)$ defined in a simply connected domain D has an antiderivative in D. (b) Show that $g(z) = \frac{1}{z(z-1)}$ has an antiderivative in the cut plane $\mathbf{C} \setminus \{x | 0 \leq x \leq 1\}$ but that $h(z) = \frac{2z-1}{z(z-1)}$ does not.

9. For $0 < p < \infty$, let $W_p[0,1]$ be the collection of measurable f on $[0,1]$ such that there is a constant $C = C_f > 0$ for which

$$|\{x \in [0,1] : |f(x)| > a\}| \leq \frac{C}{a^p}$$

for all $a > 0$. Show that $W_p[0,1] \subset L^q[0,1]$ for $0 < q < p$.

Exam Day 2 End