RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2008, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Prove that there are no simple groups of order 12.

2. Let $f(z) = \dfrac{1}{z^2 \sin z}$ for $z \in \mathbb{C}$.

(a) Determine all isolated singularities of f in the complex plane.

(b) For each singularity describe whether it is removable, a pole (give the order of the pole in this case), or essential.

(c) Compute the residue of f at each singularity.

3. Assume the functions $f_n : \mathbb{R} \to \mathbb{R}$ are continuous, for $n = 1, 2, 3, \ldots$, and that the pointwise limit

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists for every $x \in \mathbb{R}$. Fix $a \in \mathbb{R}$ and define

$$E = \{x \in \mathbb{R} \,:\, f(x) > a\}.$$

Prove that E is a countable union of closed sets.

The exam continues on the next page

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $m(E)$ and $m_e(E)$ denote Lebesgue measure and Lebesgue outer measure, respectively, for subsets E of $\mathbb{R}$. Suppose $A \subset \mathbb{R}$ is a set with the following property: there exists a number α with $0 < \alpha < 1$ such that

$$m_e(A \cap I) \leq \alpha \cdot m(I) \quad \text{for all open intervals } I \subset \mathbb{R}.$$

Prove that $m(A) = 0$. (In your proof, state clearly what properties of Lebesgue measure and outer measure you are using.)

5. Let $\mathbb{F}$ be a field (of any characteristic) and let $G = \mathrm{GL}_5(\mathbb{F})$ be the group of all invertible 5×5 matrices with entries in $\mathbb{F}$. Let $\mathcal{C}$ be the set of conjugacy classes in G and let φ be the polynomial

$$\varphi(x) = (x-1)^2(x+1).$$

Let $\mathcal{C}_\varphi$ be the set of conjugacy classes $S \in \mathcal{C}$ such that the minimal polynomial (also called the *minimum polynomial*) of each matrix in S is φ. Find representatives for each of the classes belonging to $\mathcal{C}_\varphi$. Treat the case that $\mathbb{F}$ is of characteristic two separately, noting that in this case $\varphi(x) = (x-1)^3$.

6. Let $H = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ denote the right half plane. Show that there is no holomorphic mapping $f\colon H \to H$ with $f(1) = 1$ and $f(2) = 4$.

7. Let $\mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ (also denoted as $\mathbb{Z}^{(4)}$). Consider $\mathbb{Z}^4$ as an abelian group under addition of coordinates. Let S be the subgroup of $\mathbb{Z}^4$ generated by the elements

$$(5,\, -2,\, -4,\, 1), \quad (-5,\, 4,\, 4,\, 1), \quad (0,\, 6,\, 0,\, 6).$$

Determine the structure of the abelian group Z^4/S as a direct product of cyclic groups.

The exam continues on the next page

8. Suppose that g is a bounded, Lebesgue-measurable function on $\mathbb{R}$. Define a function f on the right half plane $H = \{z \in \mathbb{C} : \text{Re } z > 0\}$ by

$$f(z) = \int_{-\infty}^{\infty} \frac{g(t)}{1 + zt^2}\, dt \qquad \text{for } z \in H.$$

(a) Prove that f is continuous on H.

(b) Prove that f is holomorphic on H.

9. Let A be an $n \times n$ matrix with complex entries a_{ij}. For $1 \le i \le n$ define

$$r_i = \sum_{j=1}^{n} |a_{ij}|.$$

Prove that if λ is an eigenvalue of A then $|\lambda| \le \max_{1 \le i \le n} r_i$.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2008, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Suppose X and Y are metric spaces and $f\colon X \to Y$ is a continuous, 1-to-1, and onto mapping.

(a) If X is compact, prove that f is a homeomorphism.

(b) If X is not compact, give an example to show that f need not be a homeomorphism.

2. Let R be a ring with identity 1 (R is not assumed to be commutative). Let I and J be two-sided ideals of R such that $R = I + J$ (the sum is not assumed to be direct). Prove that the ring $R/(I \cap J)$ is isomorphic to the direct product ring $(R/I) \times (R/J)$ (this ring is also called the *direct sum* ring and denoted as $(R/I) \oplus (R/J)$).

3. Let $\{f_n : n = 1, 2, \ldots\}$ be a sequence of functions on $[0, 1]$ that are continuous and have continuous first derivatives. Let g be a Lebesgue integrable function on $[0, 1]$. Assume that $f_n(0) = 0$ and $|f_n'(x)| \le g(x)$ for all $x \in [0, 1]$ and all n. Prove that there is a subsequence of $\{f_n\}$ that converges uniformly on $[0, 1]$.

The exam continues on the next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} (\cos x)^n\, f(x)\, dx = 0.$$

5. Suppose that V is a complex vector space of finite dimension n. Let $T : V \to V$ be a linear transformation with n distinct eigenvalues. Show that there exists a vector $v \in V$ such that the set $\{v,\, Tv,\, T^2v,\, \ldots, T^{n-1}v\}$ is a basis of V. Your proof should also describe how to find such a vector.

6. Suppose that U and V are connected open subsets of $\mathbb{C}$ and $f : U \to V$ is *biholomorphic* (a biholomorphic map is 1-to-1 and onto, and it and its inverse are both holomorphic). Which of the following statements are always true and which are sometimes false? If a statement is always true, explain why, and if a statement is sometimes false, give a specific example showing that it can be false.

(a) If U is simply connected, then V is simply connected.

(b) If U is bounded, then V is bounded.

(c) If the boundary of U is smooth (the level set of a regular value of a real-valued function on $\mathbb{C}$ with continuous partial derivatives of all orders relative to the real coordinates x, y), then the boundary of V is smooth.

7. Let D be a principal ideal domain and let E be a commutative domain containing D as a subring (a commutative domain is also called an *integral domain*). Let $a, b \in D$ and suppose that $d \in D$ is a greatest common divisor of a and b in D. Prove that d is also a greatest common divisor of a and b in E.

The exam continues on the next page

8. Suppose f is a real-valued function on $\mathbb{R}^3$ that is continuous and has continuous first partial derivatives. Let

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}(x),\ \frac{\partial f}{\partial x_2}(x),\ \frac{\partial f}{\partial x_3}(x)\right]$$

denote the gradient of f at x. Let

$$S^2 = \{x \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 + x_3^2 = 1\}$$

be the unit sphere and let g be the restriction of f to S^2. Suppose $p \in S^2$ is a point at which g has a maximum. Prove the *Lagrange multiplier condition:*

$$\nabla f(p) = \lambda \mathbf{n} \quad \text{for some scalar } \lambda,$$

where $\mathbf{n}$ is the unit outward normal vector to S^2 at p.

9. Suppose that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent series of complex numbers with sums A and B, respectively. Define $c_m = \sum_{j=0}^{m} a_j b_{m-j}$. Prove that the series $\sum_{m=1}^{\infty} c_m$ converges absolutely and that its sum is AB.

Exam Day 2 End