GRADUATE PROGRAM IN MATHEMATICS

RUTGERS UNIVERSITY

Written Qualifying Examination

Day 1, August, 2012

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3), and answer three of the six questions in Part II (numbered 4–9).

Before handing in your exam,

- Be sure your ID is on each book you are submitting.
- Label the books at the top as Book 1 of X, Book 2 of X, etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

If you work on more than three questions in Part II, indicate clearly which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day–Part I: Answer each of the following three questions.

1. Find

$$\int_0^\infty \frac{dx}{1+x^4}.$$

Justify any statement you may make about limits.

2. Let G be the abelian group generated by w, x, y, and z, subject to the defining relations $3w + 2x + 3y + 4z = 8w + 4x + 8y + 12z = 0$. Find the isomorphism type of G as a direct sum of cyclic groups.

3. Let m be Lebesgue measure on $\mathbb{R}$ and $\mathcal{F}$ a countable set of Lebesgue measurable functions defined on a Lebesgue measurable set $E \subset R$ with $m(E) < \infty$. Assume that for each $x \in E$, the set $\{|f(x)| \ : \ f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{R}$. Show that for all $\epsilon > 0$, there is a closed set $F \subseteq E$ and a number $M < \infty$ such that $m(E \cap F^c) < \epsilon$ and $|f(x)| \leq M$ for all $x \in F$ and all $f \in \mathcal{F}$.

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First Day–Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Prove that there is no one-to-one conformal map of the punctured disc $D^* = \{0 < |z| < 1\}$ onto the annulus $A = \{1 < |z| < 2\}$.

5. Let $\{f_k\}$ be a sequence of measurable functions which converge a.e. on a measurable set E to a function f. Suppose that there exist functions $\{g_k\}$ and g, all in $L^1(E)$, which satisfy $|f_k| \leq g_k$ a.e. in E, $g_k \to g$ a.e. in E, and $\int_E g_k \, dx \to \int_E g \, dx$. Show that $f \in L^1(E)$ and $\int_E f_k \, dx \to \int_E f \, dx$.

6. Let $V = \mathbb{R}^4$. Let S be the set of all 2-dimensional subspaces of V, and fix $W \in S$. Let $G = GL(V)$ act naturally on S, and let

$$H = \{g \in G \,:\, g(W) = W\}.$$

Show that H has exactly three orbits on S.

7. Suppose that (X, d) is a *complete* metric space with a finite diameter: i.e., there exists $D < \infty$ such that

$$d(x, y) \leq D \qquad \text{for all} \quad x, y \in X \ .$$

Is it true that every continuous function f on X is bounded? Prove this assertion or give a counterexample.

8. Show that $z^5 + 6z^3 - 10$ has exactly two zeros in the annulus $2 < |z| < 3$.

9. Define two bilinear forms B_1 and B_2 on $\mathbb{R}^3$ by

$$B_1(\mathbf{x}, \mathbf{y}) = 4x_1y_1 + x_2y_2 - x_3y_3 \text{ and } B_2(\mathbf{x}, \mathbf{y}) = x_1y_3 - x_2y_2 + x_3y_1,$$

where $\mathbf{x} = [x_1 \ x_2 \ x_3]$ and $\mathbf{y} = [y_1 \ y_2 \ y_3]$. Determine whether B_1 and B_2 are equivalent. Show your reasoning.

(B_1 and B_2 are called "equivalent" if and only if there exists $A \in GL_3(\mathbb{R})$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $B_1(\mathbf{x}A, \mathbf{y}A) = B_2(\mathbf{x}, \mathbf{y})$.)

Exam END

GRADUATE PROGRAM IN MATHEMATICS

RUTGERS UNIVERSITY

Written Qualifying Examination

Day 2, August, 2012

This examination will be given in two three-hour sessions, today being the second. At each session the examination has two parts.

- Answer all three of the questions in Part I (numbered 1–3), and answer three of the six questions in Part II (numbered 4–9).

Before handing in your exam,

- Be sure your ID is on each book you are submitting.
- Label the books at the top as Book 1 of X, Book 2 of X, etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

If you work on more than three questions in Part II, indicate clearly which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day–Part I: Answer each of the following three questions.

1. Let f be an entire function on $\mathbb{C}$. If the imaginary part of f is bounded on $\mathbb{C}$, prove that f must be a constant.

2. Let m be Lebesgue measure on $\mathbb{R}$. Let E be a Lebesgue measurable set such that for some number $c < 1$,

$$m(E \cap I) \leq cm(I)$$

for all open intervals I. Must it be the case that $m(E) = 0$? Either prove this, or give a counterexample.

3. Determine all groups of order $2013 = 3 \cdot 11 \cdot 61$ up to isomorphism.

The exam continues on the next page

Second Day–Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let f be a Lebesgue measurable function which is defined and finite on a Lebesgue measurable set $E \subseteq R^n$. Show that there is a Borel set $H \subseteq E$ and a Borel measurable function g defined on H such that $m(H) = m(E)$ and $g = f$ on H. Here m is Lebesgue measure.

5. Let S be an integral domain containing an element a and a subring R such that $S = R[a]$. Prove or disprove each of the following statements.

(a) If R is a principal ideal domain, then S is a principal ideal domain.

(b) If R is noetherian, then S is noetherian.

6. A polynomial $f(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$ is *symmetric* if and only if

$$f(x_1, x_2, x_3) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \text{ for all } \sigma \text{ in the symmetric group } S_3.$$

Show that every such polynomial is a polynomial in $s_1 = x_1 + x_2 + x_3$, $s_2 = x_1x_2 + x_1x_3 + x_2x_3$, and $s_3 = x_1x_2x_3$.

7. Let f be a function of bounded variation on $[0, 1]$, and define $g(x) = \max\{f(x), 1\}$ for $x \in [0, 1]$. Explain why g is differentiable a.e. on $[0, 1]$ and

$$|g'(x)| \leq |f'(x)|\,\chi_E(x) \text{ a.e. in } [0, 1], \text{ where } E = \{x \in [0, 1] : f(x) > 1\}.$$

8. Let $\mathbb{D}$ be the unit disc $|z| < 1$. Suppose that f is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$ such that $f(0) = 0$ and $f'(0) = 1$. Show that

$$|f(z) - z| \leq 2|z|^2 \text{ for all } z \in \mathbb{D}.$$

The exam continues on the next page

9. Let X and Y be locally compact metric spaces, and let $f : X \to Y$ be a continuous mapping which is bijective. Show that

$$f \text{ is a homeomorphism} \iff f^{-1}(K) \text{ is compact for all compact } K \subseteq Y.$$

Note: a metric space is locally compact if and only if every point has an open neighborhood with compact closure.

Exam END