RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August, 2014

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let V be a 5-dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation. Assume that there is $v \in V$ such that $\{v, Tv, T^2v, T^3v, T^4v\}$ spans V. Assume that the set of eigenvalues of T is precisely equal to $\{1, 2\}$. On the basis of this information, how many possible Jordan canonical forms are there for T, and what are they? Justify your answer.

2. Let $G = G_1 \times G_2$ where $G_1 \cong G_2 \cong S_4$, the symmetric group on four letters. Suppose that H is any subgroup of G such that $H \cong S_4$. Show that either $H \cap G_1 = 1$ or $H \cap G_2 = 1$.

3. Let S be an integral domain and let $a \in S$. Let R be a subring of S such that $S = R[a]$. Prove or disprove the following:

(a) If R is a principal ideal domain, then S is a principal ideal domain.

(b) If R is noetherian, then S is noetherian.

You may use major theorems in your justification as long as they are specifically mentioned.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $V = \mathbb{R}^2$. Show that the forms x_1x_2 and $2x_1^2 - 2x_2^2$ on V are equivalent.

5. Let R be a commutative ring with 1. Let I and J be ideals in R such that for every $x \in R$ there is $y \in I$ such that $x \equiv y \pmod{J}$. Show that for every $x \in R$ there is $z \in I$ such that $x \equiv z \pmod{J^2}$. (Here J^2 is the ideal generated by all products rs, $r \in J$, $s \in J$.)

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Use contour integration to show that for all $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\, dx = \pi e^{-a}.$$

Justify any limits of integrals.

2. Let $f(x)$ be a continuously differentiable real-valued function over $(-\infty, \infty)$ with $f(0) = 0$. Suppose that $|f'(x)| \leq |f(x)|$ for all $x \in (-\infty, \infty)$.

(a) Show that $f(x) = 0$ for all x in a neighborhood $(-\epsilon, \epsilon)$ of 0, for some $\epsilon > 0$.

(b) Show that $f(x) = 0$ for all $x \in (-\infty, \infty)$.

3. Let $D_1 \subset \mathbb{C}$ be the open disc centered at i with radius 1, and let $D_2 \subset \mathbb{C}$ be the open disc centered at $\frac{3}{2}i$ with radius $\frac{1}{2}$. Find an explicit biholomorphic map sending $\Omega = D_1 - \overline{D_2}$ onto the open unit disc in $\mathbb{C}$. You may express this solution as a composition of biholomorphic maps so long as each of those maps is written explicitly.

(Hint. Use the fact that the boundaries of D_1 and D_2 are tangent at $2i$.)

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^4} z^{2^n},$$

which has convergence radius 1. (Thus $f(z)$ is a well-defined holomorphic function on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.) Prove that $f(z)$ does not admit a holomorphic extension to a neighborhood of 1 in $\mathbb{C}$, that is, there does not exist a neighborhood U of 1 in $\mathbb{C}$ and a holomorphic function g defined on U such that $f|_{U \cap \Delta} = g|_{U \cap \Delta}$.

5. Write $z = x + iy$. Let $f(x, y)$ be a smooth real-valued harmonic function on the punctured unit disc $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Show that

$$f(x, y) = \mathrm{Re}(F(z)) + c \log |z|,$$

where $F(z)$ is a holomorphic function on Δ^* and $c \in \mathbb{R}$.

End of Session 2

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Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Part I. Answer all questions.

1. Let A be a subset of $[0,1]$. Let m^* be Lebesgue outer measure on $[0,1]$.
(a) State the definition of "Lebesgue measurable set".
(b) Show that A is Lebesgue measurable if and only if

$$m^*(A) + m^*(A^c) = 1,$$

where A^c is the complement of A in $[0,1]$.

2. Let f be a Lebesgue measurable function on $\mathbb{R}$ such that both

$$\int_{\mathbb{R}} |f(x)|\, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}} |xf(x)|\, dx < \infty.$$

Define the function $F(\xi)$ by

$$F(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)\, dx, \quad \xi \in \mathbb{R}.$$

Show that F is differentiable at each $\xi \in \mathbb{R}$.

3. If E_1 and E_2 are two nonempty sets in $\mathbb{R}^2$, define

$$d(E_1, E_2) = \inf_{x_1 \in E_1, x_2 \in E_2} \rho(x_1, x_2)$$

where ρ is the standard Euclidean metric.

(a) Give an example of disjoint nonempty closed sets in $\mathbb{R}^2$ with $d(E_1, E_2) = 0$.

(b) Let E_1, E_2 be nonempty sets in $\mathbb{R}^2$ with E_1 closed and E_2 compact. Show that there exist $x_1 \in E_1$ and $x_2 \in E_2$ such that $d(E_1, E_2) = \rho(x_1, x_2)$. Deduce that $d(E_1, E_2) > 0$ if such E_1, E_2 are disjoint.

Exam continues...

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Suppose that $\{f_k\}$ and $\{g_k\}$ are two sequences of functions in $L^2([0,1])$. Suppose that

$$\|f_k\|_2 \leq 1 \quad \text{for all } k \ ,$$

where the L^p norm $\|\cdot\|_p$, $1 \leq p \leq \infty$, is defined with respect to the standard Lebesgue measure μ on $[0,1]$. Suppose further that there exist $f, g \in L^2([0,1])$ such that $f_k(x) \to f(x)$ for *a.e.* $x \in [0,1]$, and that $g_k \to g$ in $L^2([0,1])$. Prove that $f_k g_k \to fg$ in $L^1([0,1])$.

5. Let f be a real-valued function on $[0,\infty)$, such that $f \in L^2([0,\infty))$. (Here the L^p spaces are defined with respect to the standard Lebesgue measure on $[0,\infty)$.) Define $F : [0,\infty) \to \mathbb{R}$ by letting $F(x) = \int_0^x f(t)\,dt$ for $x \geq 0$. Assume that $F \in L^1([0,\infty))$.

(a) Prove that $F(x)$ goes to zero as $x \to \infty$.

(b) Replace the assumption that $f \in L^2([0,\infty))$ with the assumption that $f \in L^1([0,\infty))$, and prove that also with this single change on the conditions, $F(x)$ goes to zero as $x \to \infty$.

Note: Parts (a) and (b) are independent of one another and may be solved in either order, though one can also prove both at once, use one to prove the other, etc.

End of Session 3