RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2011

Questions and Solutions

First Day—Part I: Answer each of the following three questions

1. Let f be a complex valued measurable function on $\mathbb{R}$. Let μ be the Lebesgue measure and suppose that for each $a < b$,

$$\left| \int_a^b f d\mu \right| \leq b - a.$$

Prove that $|f(x)| \leq 1$ for almost every x.

First Solution. Invoke the Lebesgue Differentiation Theorem: If x is in the Lebesgue set for f (which has complement of measure zero), then

$$f(x) = \lim_{h \to 0} \frac{1}{2h} \int_{x-h}^{x+h} f d\mu.$$

By assumption, the integral is absolute value at most $2h$, hence $|f(x)| \leq 1$.

Second Solution. Suppose not. The set $E = \{x : |f(x)| > 1\}$ is the union of the countable family of sets

$$E(\theta, \epsilon) = \{x : \mathrm{Re}\big(e^{i\theta} f(x)\big) > 1 + \epsilon\}$$

with θ, ϵ rational and $\epsilon > 0$. Thus if $\mu(E) > 0$ then there exists θ and an $\epsilon > 0$ such that the set $F = E(\theta, \epsilon)$ has positive Lebesgue measure. Let U be any open containing F such that $\mu(U) \leq (1 + \epsilon/2)\mu(F)$ (such sets exits by the outer regularity of Lebesgue measure). Then U is a countable union of disjoint intervals (a_j, b_j). Since the measure μ is countably additive, the assumption on f gives

$$\left| \int_U f \, d\mu \right| = \left| \sum_{j=1}^{\infty} \int_{a_j}^{b_j} f d\mu \right| \leq \sum_{j=1}^{\infty} (b_j - a_j) = \mu(U) \leq (1 + \epsilon/2)\mu(F)\,.$$

On the other hand, since $\operatorname{Re}(e^{i\theta} f) \geq (1+\epsilon)$ on F,

$$\left| \int_F f \, d\mu \right| = \left| \int_F e^{i\theta} f \, d\mu \right| \geq \left| \int_F \operatorname{Re}(e^{i\theta} f) \, d\mu \right| \geq (1+\epsilon)\mu(F) \,.$$

Since

$$\left| \int_U f \, d\mu \right| \geq \left| \int_F f \, d\mu \right| - \left| \int_{U \cap F^c} f \, d\mu \right| \,,$$

the last two inequalities give

$$(1+\epsilon/2)\mu(F) \geq (1+\epsilon)\mu(F) - \left| \int_{U \cap F^c} f \, d\mu \right| \,.$$

Hence $\left| \int_{U \cap F^c} f \, d\mu \right| \geq (\epsilon/2)\mu(F)$ for every such open set U containing F. Since $\mu(F) > 0$ this last inequality is a contradiction. Indeed, since f is integrable, we can choose such a U with $\int_{U \cap F^c} |f| \, d\mu < (\epsilon/2)\mu(F)$.

2. Use contour integration to evaluate

$$\int_0^\infty \frac{1}{(1+x^2)^2} \, dx.$$

Be clear about any computation of residues and about any computations of limits of integrals.

Solution. Let $f(z) = \dfrac{1}{(1+z^2)^2}$. Then the poles of f are at $z = i, -i$ and are of order 2. Let $R > 1$. and consider the counter clockwise closed path γ consisting of

$$\gamma_1 = \{y = 0, \, x \in [-R, R]\} \quad \text{and} \quad \gamma_2 = \{\operatorname{Re} z \geq 0, \, |z| = R\} \,.$$

Then by the Residue Theorem,

$$\begin{aligned} \int_\gamma f(z) dz &= 2\pi i \operatorname{Res}_f(i) = 2\pi i \, \frac{d}{dz} \Big\{ (z-i)^2 f(z) \Big\} \Big|_{z=i} \\ &= 2\pi i \Big\{ \frac{-2}{(z+i)^3} \Big\} \Big|_{z=i} = \frac{\pi}{2} \,. \end{aligned}$$

On the other hand, if $R > 2$ there is a constant $C > 0$ such that $|f(z)| \leq C/R^4$ for z on γ_2. Since the length of γ_2 is πR, this gives the estimate $\int_{\gamma_2} |f(z)dz| \leq \pi C/R^3$. Therefore

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(1+x^2)^2}\,dx = \lim_{R\to\infty}\frac{1}{2}\int_{-R}^R \frac{1}{(1+x^2)^2}\,dx$$
$$= \frac{1}{2}\lim_{R\to\infty}\left\{\int_\gamma f(z)\,dz - \int_{\gamma_2} f(z)\,dz\right\} = \frac{\pi}{4}\,.$$

3. Let S_9 denote the symmetric group on $\{1, 2, ..., 9\}$ and let $\sigma \in S_9$ be given (in table form) by

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 5 & 8 & 1 & 7 & 2 & 6 & 3 & 4 \end{bmatrix}.$$

As usual $C(\sigma)$, the centralizer of σ in S_9, is defined to be $C(\sigma) = \{\tau \in S_9 | \tau\sigma = \sigma\tau\}$. Find $|C(\sigma)|$ and justify your answer.

First Solution. In cycle form $\sigma = (194)(2576)(38)$. S_9 acts on itself by conjugation and $C(\sigma)$ is the stabilizer of σ. The orbit of σ consists of $9!(2!3!1!)/(3!4!2!) = 9!/(24)$ elements. This is because there are $9!/(3!4!2!)$ partitions of $\{1, \ldots, 9\}$ into subsets of cardinalities $3, 4$, and 2 and there are $(k-1)!$ distinct k-cycles permuting a set of k elements. Hence $|C(\sigma)| = |S_9|/|S_9\sigma| = (9!)/(9!/24) = 24$.

Second Solution. In cycle form $\sigma = (194)(2576)(38)$. S_9 acts on itself by conjugation and $C(\sigma)$ is the stabilizer of σ. If $\tau \in S_9$ then $\tau\sigma\tau^{-1} = (\tau(1)\tau(9)\tau(4))(\tau(2)\tau(5)\tau(7)\tau(6))(\tau(3)\tau(8))$. Thus if $\tau\sigma\tau^{-1} = \sigma$ then τ is determined by the choices of $\tau(1)$ (3 possibilities), $\tau(2)$ (4 possibilities), and $\tau(3)$ (2 possibilities). This gives a total of $3 \cdot 4 \cdot 2 = 24$ elements in $C(\sigma)$.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Recall that a set X in a topological space is called a G_δ set when it is a countable intersection of open sets, and it is called an F_σ set when it is a countable union of closed sets. Let μ denote Lebesgue measure on $\mathbb{R}$. Show that for every Borel set $A \subset \mathbb{R}$ there is a G_δ set G and an F_σ set F such that $F \subset A \subset G$ and $\mu(G \cap F^c) = 0$. Here $F^c = \mathbb{R} \setminus F$.

Solution. Use the inner and outer regularity of μ to get for each $k > 0$ an open set U_k and a closed set C_k such that $C_k \subset A \subset U_k$ and

$$\mu(A \cap C_k^c) \leq 1/(2k), \quad \mu(U_k \cap A^c) \leq 1/(2k)\,.$$

Without loss of generality we may assume that $U_{k+1} \subset U_k$ and $C_k \subset C_{k+1}$ for all k. Let $G = \bigcap_k U_k$, which is a G_δ set. Let $F = \bigcup_k F_k$, which is an F_σ set. Then

$$\mu(G \cap F^c) = \lim_{k\to\infty} \mu(U_k \cap F_k^c)\,.$$

Since

$$\mu(U_k \cap F_k^c) = \mu(U_k \cap A^c) + \mu(A \cap F_k^c) < 1/k\,,$$

the assertion is proved.

5. Let f be analytic on the unit disc D, and assume that $|f(z)| < 1$ for all $z \in D$. Prove that if there exist two distinct points a and b in the disc which are fixed points, that is, $f(a) = a$ and $f(b) = b$, then $f(z) = z$ for all $z \in D$.

Solution. Let $\phi_a(z) = \dfrac{a - z}{1 - \bar{a}z}$ for $z \in D$ be the Möbius transform. Then ϕ_a is an automorphism of D with inverse ϕ_{-a}. Consider $F(z) = \phi_a \circ f \circ \phi_{-a}(z)$. Then $F(0) = 0$ and

$$F\left(\frac{a-b}{1-\bar{a}b}\right) = \frac{a-b}{1-\bar{a}b} \neq 0\,. \qquad (\star)$$

Furthermore, $|F(z)| \leq 1$ for $z \in D$. By the Schwarz lemma, $F(z) = wz$ for some $w \in \partial D$. Clearly $w = 1$ by $(\star)$. Hence $f(z) = z$ for all $z \in D$.

6. Prove that there exists no simple group of order 80.

Solution. Let G be a group of order $80 = 2^4 \cdot 5$. Let n_5 be the number of 5-Sylow subgroups in G. If $n_5 = 1$ then G has a normal subgroup since all 5-Sylow subgroups are conjugate. Hence G is not simple in this case. If $n_5 > 1$ then, by the theorem on the number of Sylow subgroups, the group has 16 distinct 5-Sylow subgroups. Since these subgroups are cyclic of prime order, their pairwise intersections are trivial, so in this case G has at most $80 - 16 \cdot 4 = 16$ elements whose orders are powers of 2. These elements form only one Sylow 2-subgroup, and therefore that subgroup is normal. Hence G is not simple.

7. Let X and Y be topological spaces and $f : X \to Y$ and $g : X \to Y$ be continuous functions. Prove that if Y is a Hausdorff space then $\{x \in X : f(x) = g(x)\}$ is closed.

Solution. let $E = \{x \in X : f(x) = g(x)\}$. Suppose $z \notin E$; we show that there is a neighborhood of z disjoint from E. Since $f(z) \neq g(z)$ there are neighborhoods N_f of $f(z)$ and N_g of $g(z)$ that are disjoint. By continuity of f and g, $f^{-1}(N_f)$ and $g^{-1}(N_g)$ are neighborhoods of z. Let $N = f^{-1}(N_f) \cap g^{-1}(N_g)$. Then N is a neighborhood of z. Now we claim that $N \cap E = \emptyset$. Let $w \in N$. Then $f(w) \in N_f$ and $g(w) \in N_g$ and the disjointness of N_f and N_g implies $f(w) \neq g(w)$ so $w \notin E$.

8. Let (f_n) be a sequence of nonnegative integrable functions on $[0, 1]$ converging almost everywhere to a function $f(x)$. Prove that if

$$\lim_{n\to\infty} \int_{[0,1]} f_n d\mu = \int_{[0,1]} f \, d\mu$$

then

$$\lim_{n\to\infty} f_n = f$$

in $L^1[0, 1]$.

First Solution. Since $f_n \geq 0$, we have $0 \leq \min\{f_n, f\} \leq f$. Since $f \in L^1[0, 1]$ and $\min\{f_n, f\} \to f$ pointwise as $n \to \infty$, the Lebesgue Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_{[0,1]} \min\{f_n, f\} \, d\mu = \int_{[0,1]} f \, d\mu \, .$$

Hence the relation $|f_n - f| = f_n + f - 2 \min\{f_n, f\}$ gives

$$\lim_{n\to\infty} \int_{[0,1]} |f_n - f| \, d\mu = \lim_{n\to\infty} \int_{[0,1]} f_n \, d\mu - \int_{[0,1]} f \, d\mu = 0 \, .$$

Second Solution. Set $X = [0,1]$ and let $\epsilon > 0$ be given. Since f is integrable and nonnegative, there exists $\delta > 0$ such that $\int_E f\,d\mu < \epsilon$ for any measurable set $E \subset X$ with $\mu(E) < \delta$. Since $\mu(X) < \infty$, Egorov's Theorem implies that there exists a measurable set E with $\mu(E) < \delta$ and $\{f_n\}$ converging to f uniformly on $X \setminus E$. Since $\mu(X \setminus E) < \infty$, the uniform convergence on $X \setminus E$ and the assumed convergence of the integrals implies that there exits an integer N such that

$$\int_{X\setminus E} |f_n - f|\,d\mu < \epsilon \quad \text{and} \quad \left|\int_X (f_n - f)\,d\mu\right| < \epsilon \quad \text{for all } n \geq N\,. \qquad (\star)$$

Assume $n \geq N$. Since $f_n \geq 0$ we can use $(\star)$ to estimate

$$\begin{aligned} 0 \leq \int_E f_n\,d\mu &= \int_E (f_n - f)\,d\mu + \int_E f\,d\mu \\ &= \int_X (f_n - f)\,d\mu - \int_{X\setminus E} (f_n - f)\,d\mu + \int_E f\,d\mu \\ &\leq 3\epsilon\,. \end{aligned}$$

¿From this estimate we obtain

$$\int_E |f_n - f|\,d\mu \leq \int_E |f_n| + |f|\,d\mu \leq 4\epsilon\,.$$

Thus

$$\int_X |f_n - f|\,d\mu \leq \int_{X\setminus E} |f_n - f|\,d\mu + \int_E |f_n - f|\,d\mu \leq 5\epsilon$$

for all $n \geq N$. Since $\epsilon > 0$ was arbitrary, this proves the convergence in L^1.

9. Let A and B be commuting 8 by 8 diagonalizable matrices over the real numbers with characteristic polynomials

$$\det(A - \lambda I) = (\lambda - 1)^3(\lambda - 3)^5$$

and

$$\det(B - \lambda I) = \lambda^2(\lambda - 4)^6\,.$$

Suppose the minimum polynomial of $A - B$ is

$$(\lambda^2 - 1)(\lambda^2 - 9)\,.$$

Find the dimension of the vector space of all 8 by 8 real matrices that commute with both A and B.

Solution. Let V_1, V_3 be the eigenspaces with eigenvalues $\lambda = 1$ and $\lambda = 3$ for A. Then $\dim V_1 = 3$ and $\dim V_3 = 5$ since A is diagonalizable. Likewise, let W_0, W_4 be the eigenspaces with eigenvalues $\lambda = 0$ and $\lambda = 4$ for B. Then $\dim W_0 = 2$ and $\dim W_4 = 6$ since B is diagonalizable. Let $V_{i,j} = V_i \cap W_j$ for $i = 1, 3$ and $j = 0, 4$. Since B commutes with A,

$$\mathbb{R}^8 = V_{1,0} \oplus V_{1,4} \oplus V_{3,0} \oplus V_{3,4}$$

and A and B act by the scalars i and j, respectively, on $V_{i,j}$. We have

$$\begin{aligned} \dim V_{1,0} + \dim V_{3,0} &= 2\,, & \dim V_{1,4} + \dim V_{3,4} &= 6\,, \\ \dim V_{1,0} + \dim V_{1,4} &= 3\,, & \dim V_{3,0} + \dim V_{3,4} &= 5\,. \end{aligned} \tag{$\star$}$$

Since $A - B$ has minimum polynomial $(\lambda^2 - 1)(\lambda^2 - 9)$, the eigenvalues of $A - B$ are ± 1 and ± 3. Hence $V_{1,0}$ and $V_{3,4}$ are nonzero eigenspaces for $A - B$ with eigenvalues 1, -1, respectively. Likewise, $V_{3,0}$ and $V_{1,4}$ are nonzero eigenspaces for $A - B$ with eigenvalues 3, -3 respectively. It follows from $(\star)$ that $\dim V_{1,0} = \dim V_{3,0} = 1$, and hence $\dim V_{1,4} = 2$ and $\dim V_{3,4} = 4$.
A matrix commutes with both A and B if and only if it maps each joint eigenspace $V_{i,j}$ to itself. The action of the matrix on $V_{i,j}$ can be any linear transformation. So, the dimension of the space of all 8 by 8 real matrices that commute with both A and B is $1^2 + 2^2 + 1^2 + 4^2 = 22$.

Day 1 Exam End

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January 2011, Day 2

Second Day—Part I: Answer each of the following three questions

1. Let $f(x)$ be a function on $[0,1]$ and suppose that $f'(x)$ is defined for all $0 \le x \le 1$. Prove that $f'(x)$ is a measurable function.

Solution. Take a differentiable extension of $f(x)$ to the right of $x = 1$. For example, set $f(1+a) = f(1) + af'(1)$ for $a > 0$. Set

$$\phi_n(x) = n\big[f\big(x + \frac{1}{n}\big) - f(x)\big]\,.$$

Each $\phi_n(x)$ is continuous and therefore measurable. Since $f'(x) = \lim_{n\to\infty} \phi_n(x)$ it is measurable.

2. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie in $\{z \in \mathbb{C} : 1 \le |z| \le 2\}$.

Solution. Let $f(z) = z^7 - 5z^3 + 12$, $g(z) = z^7$, and $h(z) = 12$. On $|z| = 1$,

$$|f(z) - h(z)| = |z^7 - 5z^3| \le 6 < 12 = |h(z)|\,.$$

Therefore $f(z)$ has no zeros in $|z| < 1$ by Rouche's theorem, since $h(z)$ has no zeros there. On $|z| = 2$,

$$|f(z) - g(z)| = |5z^3 - 12| \le 5 \cdot 2^3 + 12 = 2^6 < 2^7 = |g(z)|\,.$$

Therefore $f(z)$ has 7 zeros in $|z| \le 2$ by Rouche's theorem, since $g(z)$ has 7 zeros there (counting multiplicities). By the first part, the zeros of $f(z)$ all have modulus greater than 1, and since $f(z)$ has degree 7, these are all of its zeros.

3. Are the quotient rings $\mathbb{Z}[x]/(x^3+1)$ and $\mathbb{Z}[x]/(x^3+2x^2+x+1)$ isomorphic? Provide full justification for your answer.

Solution. The rings are not isomorphic. In fact, the first ring contains zero divisors since $x^3+1=(x+1)(x^2-x+1)$ is not irreducible in $\mathbb{Z}[x]$. For the second ring, note that ± 1 is not a root of $p(x)=x^3+2x^2+x+1$. Since $p(x)$ is monic and the product of its (complex) roots is the constant term 1, it follows that $p(x)$ has no integer roots. Hence $p(x)$ is irreducible in $\mathbb{Z}[x]$, since any factorization of it would include at least one linear factor (because $p(x)$ has degree 3). Since $\mathbb{Z}[x]$ is a unique factorization domain, it follows that $p(x)$ is prime. Thus $\mathbb{Z}[x]/(p(x))$ is an integral domain. This proves that the rings are not isomorphic.

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Find the Laurent expansion of $f(z)=(1-z^2)e^{1/z}$ around $z=0$. Compute the residue at 0.

Solution. The Laurent series for $e^{1/z}$ is given by

$$e^{1/z}=\sum_{n=0}^{\infty}\frac{1}{n!z^n}\,.$$

Hence

$$f(z)=\sum_{n=0}^{\infty}\left(\frac{1}{n!z^n}-\frac{1}{n!z^{n-2}}\right)=-z^2-z+\sum_{n=0}^{\infty}\frac{n^2+3n+1}{(n+2)!}z^{-n}\,.$$

The coefficient of z^{-1} in the series is $(1^2+3\cdot 1+1)/3!$. Thus $\mathrm{Res}_f(0)=5/6$.

5. Let f be a complex-valued measurable function on $\mathbb{R}$. Let μ be Lebesgue measure and suppose that for each $g\in L^2(\mu)$, the function $fg\in L^1(\mu)$. Show that $f\in L^2(\mu)$.

Solution. For each positive integer N define

$$F_N=\{g\in L^2(\mu)\,:\,\int|fg|\,d\mu\le N\}\,.$$

The assumption on f is that

$$L^2(\mu) = \bigcup_{N=1}^{\infty} F_N\,. \tag{$\star$}$$

We first show that F_N is closed in $L^2(\mu)$. Indeed, if $\{g_j\}$ is a sequence in F_N that converges to a function g in the $L^2(\mu)$ norm, then by passing to a subsequence we may assume that $\{g_j\}$ that converges to g almost everywhere. Since fg_j converges to fg almost everywhere, Fatou's Lemma implies that

$$\int |fg|\,d\mu \le \liminf_{j\to\infty} \int |fg_j|\,d\mu \le N\,.$$

Since $L^2(\mu)$ is a complete metric space, Baire's Theorem asserts that one of the sets on the right side of ($\star$) must have an interior point. Hence there exists an N, a function $g_0 \in F_N$, and a real number $r > 0$ such that the ball of radius r around g_0 is contained in F_N. Thus for all unit vectors $h \in L^2(\mu)$,

$$r\int |hf|d\mu \le \int |(rh+g_0)f|d\mu + \int |g_0 f|d\mu \le N + \int |g_0 f|d\mu\,.$$

This proves that

$$M = \sup_{\|h\|_2=1} \int |hf|d\mu < \infty\,. \tag{$\star\star$}$$

Hence the linear functional $g \mapsto F(g) = \int fg\,d\mu$, for $g \in L^2(\mu)$, is bounded with bound M. By the Riesz Representation Theorem, there exists a function $\varphi \in L^2(\mu)$ such that $F(g) = \int \varphi g\,d\mu$ for all $g \in L^2(\mu)$. Hence $f = \varphi$ almost everywhere, so $f \in L^2(\mu)$.

($\star\star$) can also be proved without the help of the Baire Category Theorem as follows. First one may assume that $f \ge 0$, as the assumption on f is also valid for the real and imaginary parts of f, and then for their positive and negative parts. If ($\star\star$) is not valid, then for any $k \in \mathbb{N}$, there exists g_k with $||g_k||_{L^2(\mu)} = 1$ such that $\int fg_k\ d\mu = a_k \ge k$. One may even take $g_k \ge 0$. Define

$$h_l = \sum_{k=1}^{l} (ka_k)^{-1} g_k,$$

then $||h_l||_{L^2(\mu)} \le \sum_{k=1}^{l} k^{-2}$, and $\int fh_l\ d\mu = \sum_{k=1}^{l} k^{-1}$. It now follows by Monotone Convergence Theorem that

$$h = \sum_{k=1}^{\infty} (ka_k)^{-1} g_k \in L^2(\mu),$$

but $\int fh d\mu \geq \int fh_l \, d\mu$ for all l, which leads to $\int fg d\mu = \infty$, a contradiction.

6. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be vectors over the field $F = \mathbb{Z}/3$. Show that the bilinear forms $B(x, y) = -x_1y_1 - x_2y_2$ and $D(x, y) = x_1y_1 + x_2y_2$ are equivalent.

Solution. Interpreting x and y as column vectors, then $D(x, y) = x^t y$ and $B(x, y) = x^t(-I)y$. Thus we need a matrix

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with coefficients in F such that $P^t(-I)P = I$, or $P^tP = -I$. Thus entries of P must satisfy $a^2 + c^2 = -1$, $b^2 + d^2 = -1$, and $ab + cd = 0$. One solution is $a = b = d = 1$ and $c = -1$, since $2 = -1$ in F.

7. Consider the curve $S = \{(x, \sin(1/x)) \;:\; x \in (0, 1]\} \subseteq \mathbb{R}^2$. Let $T = S \cup (\{0\} \times [-1, 1])$. Show that T is a connected subset of $\mathbb{R}^2$.

Solution. Suppose A, B is a pair of disjoint non-empty open sets of $\mathbb{R}^2$ whose union contains S. S is the image of the connected set $(0, 1]$ under the continuous map $f : (0, 1] \longrightarrow \mathbb{R}^2$ given by $f(x) = (x, \sin(1/x))$ and is hence connected. Thus S is a subset of A or B; assume $S \subset A$. Every point $(0, y)$ of $\{0\} \times [-1, 1] = T - S$ is a limit point of S. Indeed, let $b = \arcsin(y)$ and for k a positive integer let $x_k = 1/(b + 2k\pi)$. Then $f(x_k) = (x_k, y)$, which converges to $(0, y)$ as $k \to \infty$. So if $(0, y) \in B$ then B contains points of S since B is open. This contradicts the assumption $A \cap B = \emptyset$, so we conclude that $T \subseteq A$, and hence T is connected.

8. Let G be a finite group. Prove that G is cyclic if and only if G has exactly one subgroup of order n for each positive integer n dividing $|G|$.

Solution. Let $N = |G|$. If G is a cyclic group then $G \cong \mathbb{Z}/N\mathbb{Z}$. The subgroups of G are in one-to-one correspondence with subgroups $k\mathbb{Z}$ of $\mathbb{Z}$ containing $N\mathbb{Z}$, i.e., with $k \mid N$, by the isomorphism theorems. Since $k\mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/(N/k)\mathbb{Z}$ there is exactly one subgroup of order $n = N/k$ for each n dividing N.

Conversely, suppose that G has a unique subgroup G_n of order n for each n dividing N. We proceed by induction on N. Let p be a prime dividing N. By Cauchy's Theorem every G_n with p dividing n has a subgroup of order p, so $G_p \leq G_n$ for all such n. Therefore by the isomorphism theorems, G/G_p has a unique subgroup of each order dividing N/p. By induction G/G_p is cyclic. Choose $x \in G$ such that $G_p x$ generates G/G_p. Then N/p divides the order of x.

If x has order N, then G is cyclic. So assume that x has order N/p. Also $\langle x \rangle G_p = G$ so $N = |G| = |\langle x \rangle||G_p|/|\langle x \rangle \cap G_p| = (N/p)p/|\langle x \rangle \cap G_p|$. Therefore $\langle x \rangle \cap G_p = 1$. Every subgroup H of G is normal, since H is the only subgroup of order $|H|$. Therefore, $G = \langle x \rangle \times G_p$. If p divides N/p, then $\langle x \rangle$ has a subgroup of order p, contradicting the uniqueness of G_p. Therefore p does not divide N/p, so $G \cong \mathbb{Z}_{N/p} \times \mathbb{Z}_p \cong \mathbb{Z}_N$ by the Chinese Remainder Theorem.

9. Exhibit a conformal map $f : U \to D$, (that is, a bijective map f from U to D, such that both f and its inverse are holomorphic), where D is the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and U is the set $\{z \in D : \text{Re } z > 0\}$.

Solution. First we let

$$g(z) = \frac{i - z}{i + z} \,.$$

Since fractional linear transformations carry circles (a line being a circle through ∞) to circles, and g has the values $g(i) = 0$, $g(0) = 1$, $g(-i) = \infty$, and $g(1) = i$, it follows that g maps the imaginary axis to the real axis, and maps the unit circle to the imaginary axis. Also

$$g(iy) = \frac{1 - y}{1 + y} \,,$$

so g maps $\{iy : -1 < y < 1\}$ to the positive real axis. Since $1/2 \in U$ and $g(1/2) = (3 + 4i)/5$ is in the first quadrant, it follows that g maps U conformally to the first quadrant

$$Q = \{x + iy \in \mathbb{C} : x > 0 \text{ and } y > 0\} \,.$$

Next, the map $z \mapsto z^2$ sends Q conformally to the upper half-plane

$$H = \{x + iy \in \mathbb{C} : y > 0\} \,.$$

Finally, if we let

$$h(z) = \frac{z - i}{z + i}\,,$$

then h maps H to D, because $|h(z)| = 1$ if z is real (since in that case $h(z) = w/\bar{w}$ if $w = z - i$), and $h(i) = 0$. So, the composite map f given by

$$f(z) = h(g(z)^2) = \left\{ \left(\frac{i - z}{i + z}\right)^2 - i \right\} \Big/ \left\{ \left(\frac{i - z}{i + z}\right)^2 + i \right\}$$

sends U conformally onto D.

Remark. It is recommended that maps constructed in this problem be illustrated with appropriate sketches. Relevant points and boundaries in the domains and ranges should be labeled.

Exam Day 2 End