RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

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Session I. Algebra

This examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

- Answer **all** of the questions in Part I (numbered 1, 2, 3).
- Answer **one** of the questions in Part II (numbered 4, 5). If you work on both optional questions in Part II, indicate **clearly** which one should be graded. No additional credit will be given for more than one of the optional questions. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that don’t want graded is crossed out or clearly labeled to be ignored.

Session I–Part 1: Answer each of the following three questions

1. Prove that there is no simple group of order 80 January 2011

2. Let $\mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, considered as an abelian group under addition of coordinates. Let S be the subgroup of $\mathbb{Z}^4$ generated by the elements January 2008

$$(5, -2, -4, 1), \quad (-5, 4, 4, 1), \quad (0, 6, 0, 6)$$

Determine the structure of the abelian group $\mathbb{Z}^4/S$ as a direct product of cyclic groups.

3. Suppose A is a 5×5 complex matrix and $(A - 2I)^5 = 0$. January 2012

(a) What Jordan canonical forms are possible for A?

(b) Suppose that there exists another 5×5 complex matrix B such that $AB = BA$ and the minimal polynomial of B is $t^3 + t$. Now which of your answers to (a) are still possible Jordan canonical forms for A? Explain your reasoning.

Session I–Part 2: Answer one of the following questions. If you work on more than one question, indicate clearly which one should be graded.

4. Let S_9 denote the symmetric group on $\{1, 2, \ldots, 9\}$ and let $\sigma \in S_9$ be given (in table form) by January 2011

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 5 & 8 & 1 & 7 & 2 & 6 & 3 & 4 \end{bmatrix}$$

Let $C(\sigma) = \{\tau \in S_9 : \tau\sigma = \sigma\tau\}$ denote the centralizer of σ in S_9. Find the order of $|C(\sigma)|$ and justify your answer.

5. Let D be a principal ideal domain and let E be a commutative domain containing D as a subring (a commutative domain is also called an *integral domain*). Let $a, b \in D$ and suppose that $d \in D$ is a greatest common divisor of a and b in D. Prove that d is also a greatest common divisor of a and b in E. January 2008

Session I End

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Sesson II. Complex Analysis and Advanced Calculus

This examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

- Answer **all** of the questions in Part I (numbered 1, 2, 3).
- Answer **one** of the questions in Part II (numbered 4, 5). If you work on both optional questions in Part II, indicate **clearly** which one should be graded. No additional credit will be given for more than one of the optional questions. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Session II–Part 1: Answer each of the following three questions

1. Consider the power series $f(z) = \sum_{n=1}^{\infty} \sqrt{n}\, z^n$. August 2013

(a) Show that this formula defines $f(z)$ as a holomorphic function in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

(b) Show that f does not extend to a continuous complex-valued function on the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$.

2. Use contour integration to evaluate January 2011

$$\int_0^\infty \frac{1}{(1+x^2)^2}\, dx\,.$$

Give details of computations of residues and justify any limits of integrals.

3. Find a biholomorphic mapping (i.e., a holomorphic mapping $S \to D$ with a holomorphic inverse) from the sector August 2013

$$S = \{re^{i\theta} : 0 < r < 1 \text{ and } 0 < \theta < \frac{\pi}{3}\}$$

onto the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Describe the mapping as a composition of rational functions and elementary transcendental functions, and show by appropriate sketches relevant points and boundaries in the domains and ranges of these functions.

Session I–Part 2: Answer one of the following questions. If you work on more than one question, indicate clearly which one should be graded.

4. Let $f(z) = 6z^5 - 10z^3 - 2z - 1$. Find the number of zeros (counted with multiplicity) of f in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. August 2009

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function that satisfies $f(0) = 0$ and $f'(0) > 0$ and $f''(x) \geq f(x)$ for all $x \geq 0$. Prove that $f(x) > 0$ for all $x > 0$. January 2009

Session II End

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Session III. Real Analysis and Elementary Point Set Topology

This examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

- Answer **all** of the questions in Part I (numbered 1, 2, 3).
- Answer **one** of the questions in Part II (numbered 4, 5). If you work on both optional questions in Part II, indicate **clearly** which one should be graded. No additional credit will be given for more than one of the optional questions. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Session III–Part 1: Answer each of the following three questions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative Lebesgue measurable function such that the Lebesgue integral $\int_a^b f(x)\,\mathrm{d}x < \infty$ for all real numbers $-\infty < a < b < \infty$. For every Lebesgue measurable set $A \subset \mathbb{R}$ define August 2009

$$\mu(A) = \int_A f(x)\,\mathrm{d}x$$

(a) Prove that μ is a σ-finite measure on the Lebesgue measurable sets of $\mathbb{R}$.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Prove that

$$\int_{\mathbb{R}} g(x)\,d\mu(x) = \int_{\mathbb{R}} g(x)f(x)\,dx,$$

in the sense that when either integral exists (in the sense of Lebesgue integration theory) then the other integral also exists, and they are equal.

2. For $n = 1, 2, 3, \ldots$ let $f_n : [0, 1] \to [0, \infty)$ be a Lebesgue integrable function. Assume that August 2008

$$\int_0^1 f_n(x)\,\mathrm{d}x = 1 \text{ and } \int_{1/n}^1 f_n(x)\,\mathrm{d}x < \frac{1}{n} \quad \text{for all } n.$$

For $0 \leq x \leq 1$ define $g(x) = \sup\{f_n(x) : n = 1, 2, \ldots\}$. Prove that

$$\int_0^1 g(x)\,\mathrm{d}x = \infty$$

3. Let X be a compact metric space and let $\mathcal{F}$ be any nonempty set of real valued functions on X that is uniformly bounded and equicontinuous. Is the function $g(x) = \sup\{ f(x) : f \in \mathcal{F} \}$ necessarily continuous? Justify your answer. August 2011

The exam continues on the next page

Session III–Part 2: Answer one of the following questions. If you work on more than one question, indicate clearly which one should be graded.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Assume that January 2008

$$\int_{\mathbb{R}} |f(x)| \, \mathrm{d}x < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} (\cos x)^n \, f(x) \, \mathrm{d}x = 0.$$

5. Suppose that $g(x)$ is a continuous function on the interval August 2010

$$I = \{x \in \mathbb{R} \, : \, 0 \leq .x \leq 1\}.$$

Define functions $f_n(x)$ on I by

$$f_0(x) = g(x) \quad \text{and} \quad f_n(x) = \int_0^x f_{n-1}(t) \, dt \quad \text{for } n = 1, 2, \ldots..$$

Prove that the sequence $\{f_n(x)\}$ converges to 0 uniformly on I.

Session III End