Discrete Fourier and Wavelet Transforms: Mathematical Microscopes for Signal Processing

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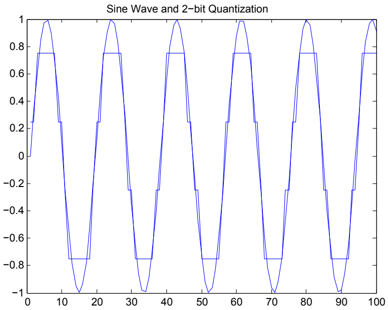
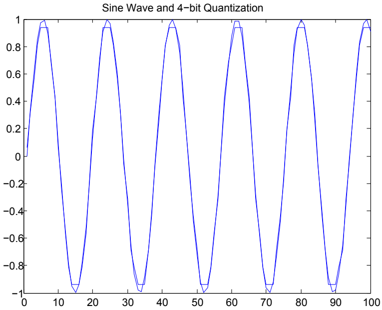
analog signal $f(t)$ $(a \le t \le b) \longrightarrow$ digital signal $\mathbf{v}$

$\mathbf{v} = Q[f(t_0), f(t_1), \ldots, f(t_{N-1})]^{\mathrm{T}}$ $\quad Q =$ quantize entries (# bits)

$t_j = a + j\Delta t$ with $\Delta t = (b-a)/N$ $\quad$ (sample at rate N)





more bits gives better sound but needs more memory space

Linear algebra model:

digital signal $\longleftrightarrow$ vector in complex vector space $\cong \mathbb{C}^N$

Fix integer $N \geq 2$ $\quad \ell^2[\mathbb{Z}/N\mathbb{Z}] =$ vector space of N-periodic signals

$\phi : \mathbb{Z} \to \mathbb{C}$ with $\phi(k+N) = \phi(k)$ for all $k \in \mathbb{Z}$

inner product $\quad \langle \phi, \psi \rangle = \sum_{k=0}^{N-1} \overline{\phi(k)}\psi(k) \quad$ ($\bar{z} =$ complex conj.)

$\ell^2[\mathbb{Z}/N\mathbb{Z}] \longleftrightarrow \mathbb{C}^N$ by $\phi \longleftrightarrow x = [\phi(0), \cdots, \phi(N-1)]^{\mathrm{T}}$

Transform Method

Choose invertible matrix $A = [v_0, \cdots, v_{N-1}]$ $\quad$ (basis for $\mathbb{C}^N$)

Let $v_0^*, \ldots, v_{N-1}^*$ be rows of A^{-1} $\quad$ (dual basis)

A-transform $X = A^{-1}x = [c_0, \cdots, c_{N-1}]^{\mathrm{T}}$, entries $c_k = v_k^* x$

Inversion formula $\quad (\star) \quad x = AX = c_0 v_0 + \cdots + c_{N-1} v_{N-1}$

Example $\quad A^{-1} = \overline{A}^T$ (Orthonormal basis) $\quad \Longleftrightarrow v_k^* = \overline{v}_k^T$

$\Longleftrightarrow$ energy preserved: $(x, x) = (X, X) = |c_0|^2 + \cdots + |c_{N-1}|^2$

Transform Problem

Find the best transform for expanding a particular type of signal x

- Large percentage of the coefficients c_k in $(\star)$ are small
- Replace small c_k by 0 to get compressed signal $\tilde{x}$ with small percentage of nonzero coefficients (good approximation to x)

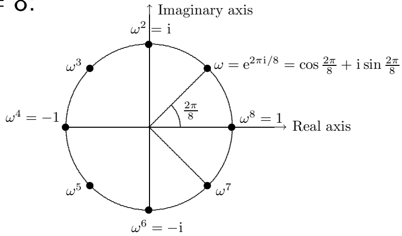
Shift operator For $\phi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$ $\quad S\phi(k) = \phi(k-1)$ $\quad (S^N = I)$

Problem Find eigenvectors for S

Let $\omega = \mathrm{e}^{2\pi\mathrm{i}/N}$ $\quad (\mathrm{i} = \sqrt{-1})$ $\quad$ Then $\omega^N = 1$ and

$1, \omega, \omega^2, \ldots, \omega^{N-1}$ are the solutions to $z^N = 1$ (roots of unity)

$N = 8$:



Set $\phi_p(k) = \omega^{kp}$ for $k, p = 0, \ldots, N-1$ $\quad$ Then $\phi_p \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$

$\phi_p =$ discrete sample of continuous wave f_p (frequency p):

$$f_p(t) = \mathrm{e}^{2\pi p t \mathrm{i}} = \cos(2\pi p t) + \mathrm{i}\sin(2\pi p t) \qquad (t = k/N)$$

$$\phi_p \longleftrightarrow \begin{cases} \text{low freq. wave when } p \approx 0 \pmod N & \omega^p \approx 1 \\ \text{high freq. wave when } p \approx N/2 \pmod N & \omega^p \approx -1 \end{cases}$$

Fourier Basis $\{\phi_0, \phi_1, \dots, \phi_{N-1}\}$ for $\ell^2[\mathbb{Z}/N\mathbb{Z}]$

- Eigenvectors for S: $S\phi_p = \omega^{-p}\phi_p$ (eigenvalue $= \omega^{-p}$)
- Orthogonality: $\langle \phi_p, \phi_q \rangle = \begin{cases} N & \text{if } p \equiv q \mod N \\ 0 & \text{else} \end{cases}$

Discrete Fourier transform (DFT) of $\phi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$

$$\widehat{\phi}(p) = \langle \phi_p, \phi \rangle = \sum_{k=0}^{N-1} \omega^{-kp}\, \phi(k) \quad \text{for } \phi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$$

Fourier synthesis: $\phi = (1/N)\{\widehat{\phi}(0)\,\phi_0 + \cdots + \widehat{\phi}(N-1)\,\phi_{N-1}\}$

Matrix Description (calculated by FFT "Fast Fourier Transform")

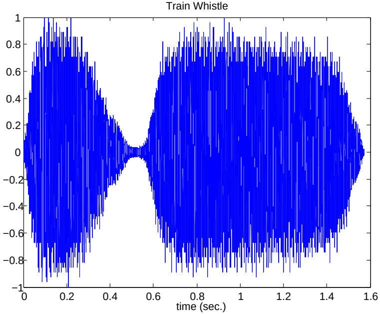
$\phi_p \longleftrightarrow E_p \in \mathbb{C}^N$ $(p = 0, \dots, N-1)$ Fourier basis for $\mathbb{C}^N$

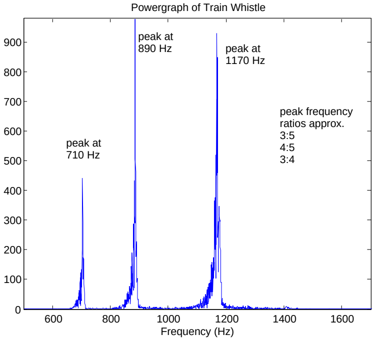
Fourier matrix $F_N = [\overline{E_0}, \cdots, \overline{E_{N-1}}]$ (j, k entry $\omega^{-(j-1)(k-1)}$)

$$N = 4\colon\ F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix}$$

If $\phi \longleftrightarrow y \in \mathbb{C}^N$ then $\widehat{\phi} \longleftrightarrow Y = F_N y$ and $y = (1/N)\overline{F}_N Y$

Frequency Analysis



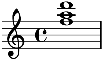


frequency peaks near 710Hz, 890Hz, 1170Hz $(N = 12,880)$

analog signal to synthesize peak frequencies

$f(t) = 0.45\sin(2\pi * 710t) + 0.95\sin(2\pi * 890t) + 0.93\sin(2\pi * 1170t)$

What is missing from this model?



(quarter note, rest, half note) d minor chord (whole note)

Need time & frequency information!

Given: Digital signal $s_k = \begin{bmatrix} s_k(0) \\ s_k(1) \\ \vdots \\ s_k(N-1) \end{bmatrix}$ length $N = 2^k$ (level k)

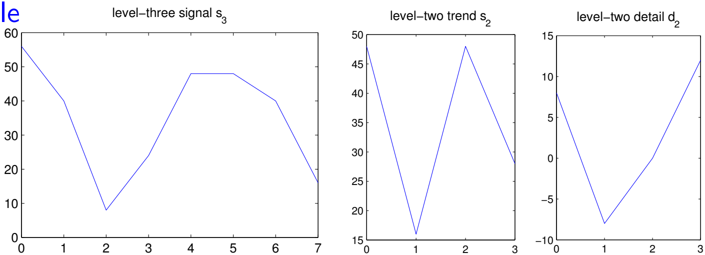
Haar Wavelet Transform (pyramid algorithm)

- Downsample $s_k \longrightarrow \begin{bmatrix} (s_k)_{\text{even}} \\ (s_k)_{\text{odd}} \end{bmatrix}$ with

$$(s_k)_{\text{even}} = \begin{bmatrix} s_k(0) \\ s_k(2) \\ \vdots \\ s_k(N-2) \end{bmatrix}, \ (s_k)_{\text{odd}} = \begin{bmatrix} s_k(1) \\ s_k(3) \\ \vdots \\ s_k(N-1) \end{bmatrix} \ (\text{length } N/2)$$

- Trend (level $k-1$) $s_{k-1} = \frac{1}{2}\left\{(s_k)_{\text{even}} + (s_k)_{\text{odd}}\right\}$
- Detail (level $k-1$) $d_{k-1} = \frac{1}{2}\left\{(s_k)_{\text{even}} - (s_k)_{\text{odd}}\right\}$
- Iterate on trend $s_{k-1} \longrightarrow [s_{k-2},\ d_{k-2}], \quad s_{k-2} \longrightarrow \cdots$

Example

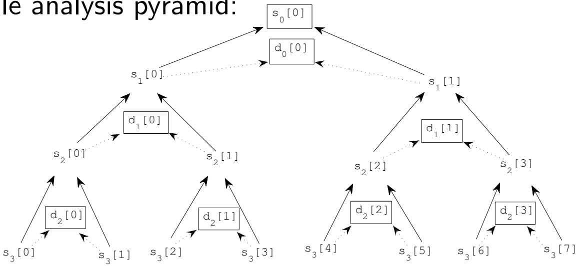


Three-scale analysis pyramid:

level 0

level 1

level 2



Three-scale multiresolution analysis $s_3 \longrightarrow [s_0, d_0, d_1, d_2]$

Three-scale multiresolution synthesis $s_3 = W^{(3)} \begin{bmatrix} s_0 \\ d_0 \\ d_1 \\ d_2 \end{bmatrix}$

Haar three-scale synthesis matrix (8×8):

$W^{(3)} = [u_0,\ v_0,\ v_1,\ S^4 v_1,\ v_2,\ S^2 v_2,\ S^4 v_2\ S^6 v_2]$ (S = shift)

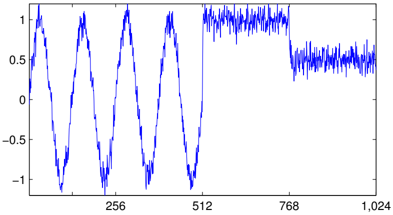
$$u_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \quad v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

trend   coarse detail   medium detail   fine detail

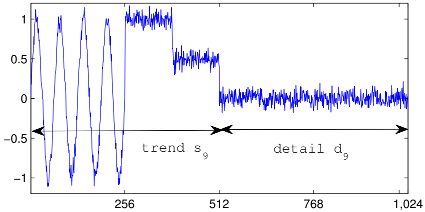
Differences between detail vectors for Haar basis and Fourier basis:

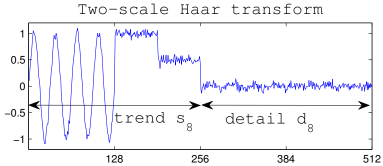
- localized in time (multiple time scales)
- time shifted and rescaled samples of one function (wavelet)

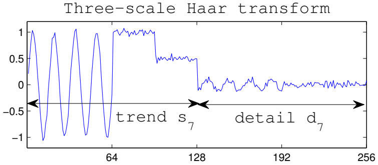
Wave + Step Function + Noise



One-scale Haar transform



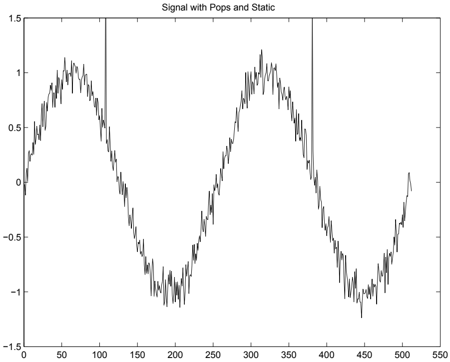


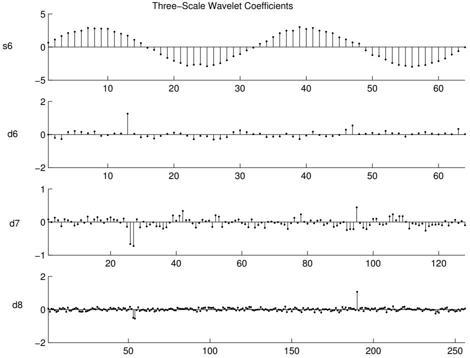


Pyramid algorithm: $x \longrightarrow [s_7, d_7, d_8, d_9] \quad (N = 2^{10})$

There are many other mathematical microscopes!

Three-scale LeGall wavelet transform
(used in JPEG 2000 lossless image compression algorithm)
signal $\longrightarrow [s_6, d_6, d_7, d_8] \quad (N = 2^9 = 512)$





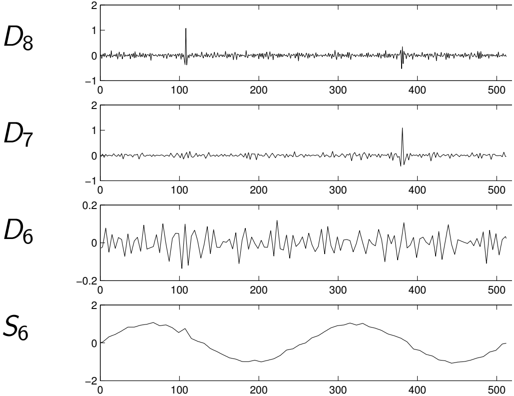
- Trend s_6 shows the low frequency signal content
- Time location of pops is clear at each level of detail
- Most of the detail coefficients are small (noise)

Three level wavelet synthesis of signal:

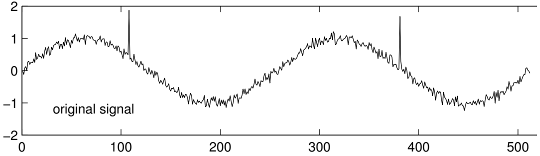
Let D_j = inverse wavelet transform of details d_j ($j = 8, 7, 6$)

S_6 = inverse wavelet transform of trend s_6

Then original signal $= D_8 + D_7 + D_6 + S_6$ (each term in $\mathbb{C}^{512}$)



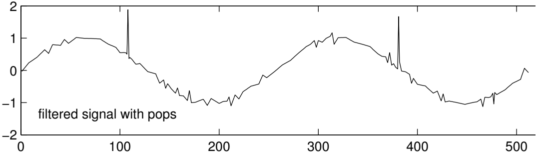
Wavelet Filtering and Compression



Threshold Compression

- 97% of detail coefficients in d_6, d_7 and d_8 are less than 0.2.
- Replace each such small coefficient by 0 to get $\tilde{d}_6$, $\tilde{d}_7$, $\tilde{d}_8$
- Calculate inverse wavelet transforms $\widetilde{D}_6$, $\widetilde{D}_7$, $\widetilde{D}_8$ and add to S_6

$S_6 + \widetilde{D}_6 + \widetilde{D}_7 + \widetilde{D}_8$:



Noise filtered out but pops still in (6 : 1 compression of signal)
(Not possible using Fourier basis)

W = one-scale wavelet analysis matrix

X = image matrix (256 $\times$ 256 eight-bit matrix)

WXW^{T} = wavelet transform (partitioned matrix)

Original Lena Image

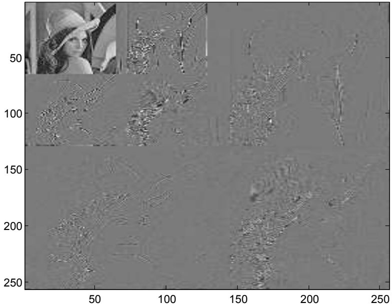
One-scale Wavelet Transform

trend 128×128	vertical details
horizontal details	diagonal details

Multiscale Image Transforms and Edge Detection

Pyramid algorithm for two-scale transform of image matrix

- Keep the 3 one-scale detail matrices
- Make a wavelet transform of the trend matrix



Two-scale wavelet transform

Inverse transform of details (omit level-two trend)

Famous Cartoon (mathematician to engineer after seeing machine)

"It works in practice, but does it work in theory?"

Problem How do we build wavelet transforms?

Given: signal ϕ and polynomial $f(z) = a_0 + a_1 z + \cdots + a_{N-1} z^{N-1}$
Form moving average of the signal (as for Haar trend and detail)

$$a_0\phi(k) + a_1\phi(k-1) + \cdots + a_{N-1}\phi(k-N+1) = (f(S)\phi)(k)$$

where $S =$ shift operator and $f(S) = a_0 + a_1 S + \cdots + a_{N-1} S^{N-1}$

Definition The linear transformation $f(S)$ on $\ell^2[\mathbb{Z}/N\mathbb{Z}]$ is called a filter, and $f(S)\phi$ is the filtered signal

Example On Fourier basis $S\phi_p = \omega^{-p}\phi_p$ $\;(\omega = \mathrm{e}^{2\pi\mathrm{i}/N})$ Hence

$$f(S)\phi_p = (a_0 + a_1\omega^{-p} + \cdots + a_{N-1}\omega^{-p(N-1)})\phi_p = f(\omega^{-p})\phi_p$$

- ϕ_p is an eigenvector for $f(S)$ with eigenvalue $\lambda = f(\omega^{-p})$
- If $\phi = \sum_{p=0}^{N-1} c_p\phi_p$ then $f(S)\phi = \sum_{p=0}^{N-1} f(\omega^{-p}) c_p \phi_p$
- Low pass filter: $f(S)$ attenuates high frequencies ($p \approx N/2$)
  $\iff f(-1) = 0 \iff f(z) = (z+1)^m g(z) \quad (m \geq 1)$
- High pass filter: $f(S)$ attenuates low frequencies ($p \approx 0$)
  $\iff f(1) = 0 \iff f(z) = (z-1)^m g(z) \quad (m \geq 1)$

Filter Banks and Discrete Wavelet Transforms

Construct a wavelet analysis transform of $\phi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$ (N even) using lowpass and highpass filters followed by downsampling

- Choose a polynomial $h_0(z)$ with $h_0(-1) = 0$ to get a lowpass filter $H_0 = h_0(S)$ and calculate $H_0\phi$ (length N)
- Choose a polynomial $h_1(z)$ with $h_1(1) = 0$ to get a highpass filter $H_1 = h_1(S)$ and calculate $H_1\phi$ (length N)
- Introduce downsampling operator D on $\psi \in \ell^2[\mathbb{Z}/N\mathbb{Z}]$:
  $$(D\psi)(k) = \psi(2k) \quad (D\psi \text{ period } N/2), \quad \psi \longleftrightarrow v \in \mathbb{C}^N$$
  $$\ell^2[\mathbb{Z}/N\mathbb{Z}] \xrightarrow{D} \ell^2[\mathbb{Z}/(N/2)\mathbb{Z}], \quad D\psi \longleftrightarrow v_{\mathrm{even}} \in \mathbb{C}^{N/2}$$
- Calculate trend $= DH_0\phi$ and detail $= DH_1\phi$ (lengths $= N/2$)
- Define a wavelet transform $W\phi = \begin{bmatrix} \text{trend}(\phi) \\ \text{detail}(\phi) \end{bmatrix}$

Important: length(ϕ) = length($W\phi$), so $W \longleftrightarrow N \times N$ matrix

Perfect Reconstruction (PR) Problem   Is W invertible?

Energy Preservation Problem   Does W preserve the energy of a signal? (Important for signal compression)

PR Low Pass & High Pass Filter Pairs

Algebraic condition for PR

$(\star\star)$ $\quad h_0(z)h_1(-z) - h_0(-z)h_1(z)$ is a nonzero monomial

Examples

- Haar transform

$$h_0(z) = 1 + z \quad \text{and} \quad h_1(z) = 1 - z$$

Here $\quad h_0(z)h_1(-z) - h_0(-z)h_1(z) = (1+z)^2 - (1-z)^2 = 4z$

- LeGall transform (= CDF(2,2) transform)

$$h_0(z) = (1+z)^2(1 - 4z + z^2) \quad \text{and} \quad h_1(z) = (1-z)^2$$

- CDF(p,q) transforms (Ingrid Daubechies & collaborators, 1992)

Given positive integers p, q with $p + q$ even, there exits $g(z)$ so

$$h_0(z) = (1+z)^q g(z) \quad \text{and} \quad h_1(z) = (1-z)^p$$

satisfy $(\star\star)$ (explicit formula for $g(z)$ with binomial coefficients)

- CDF(9,7) filters used in JPEG 2000 image compression algorithm

In these examples only the Haar transform preserves energy

Theorem

Suppose the polynomial $h_0(z)$ satisfies $h_0(-1) = 0$ and

$$(\sharp) \qquad h_0(z)h_0(z^{-1}) + h_0(-z)h_0(-z^{-1}) = 2$$

Define $h_1(z) = zh_0(-z^{-1})$ and let $H_0 = h_0(S)$, $H_1 = h_1(S)$. Then H_0, H_1 are the low pass & high pass filters for an energy-preserving PR wavelet transform.

Remark Equation $(\sharp) \iff$ system of quadratic equations for the coefficients of $h_0(z)$

Examples

- Haar transform (normalized) Take $h_0(z) = (1 + z)/\sqrt{2}$ Then $(\sharp) = \frac{1}{2}\{(1 + z)(1 + z^{-1}) + (1 - z)(1 - z^{-1})\} = 2$
- Daub4 transform Take $h_0(z) = (a + bz + cz^2 + dz^3)/\sqrt{32}$ where $a = 1 + \sqrt{3}$, $b = 3 + \sqrt{3}$, $c = 3 - \sqrt{3}$, $d = 1 - \sqrt{3}$ Then $h(-1) = a - b + c - d = 0$ and $(\sharp)$ is satisfied
- Daub2k transform (any positive integer k) Find complex roots of polynomial used in CDF(k,k) transform (No explicit formula)