RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2022

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let G be a finite simple group. Prove that $G \times G$ has exactly 4 normal subgroups (including $G \times G$) if and only if G is non-abelian.

2. Let R be a principal ideal domain and I and J be ideals of R. Show that $I \cap J = IJ$ holds if and only if $I = 0$ or $J = 0$ or $I + J = R$.

3. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix with real coefficients. Show that all eigenvalues of A are non-negative if and only if $A = P^T P$ for some matrix $P \in M_n(\mathbb{R})$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let R be an integral domain and $R[x, y, z]$ the polynomial ring in three variables over R. Show that $I = \langle x^3 - y^2, y^3 - z^2 \rangle \subset R[x, y, z]$ is a prime ideal. Hint: Show that I is the kernel of a ring homomorphism $R[x, y, z] \to R[t]$.

5. Let A and B be commuting complex matrices. Assume that $B \notin \mathbb{C}[A]$, that is, B cannot be written as a polynomial in A. Show that some eigenspace of A has dimension at least two.

End of Session on Algebra

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Written Qualifying Examination

Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. Show that $u(x, y) = \ln(x^2 + y^2)$ is a harmonic function in $\mathbb{C} \setminus \{0\}$. Find a conjugate harmonic function of $u(x, y)$ in $\mathbb{C} \setminus \{x : x \leq 0\}$. Show that it does not have a conjugate harmonic function in $\mathbb{C} \setminus \{0\}$.

2. Evaluate the following integral

$$\int_{-\infty}^{+\infty} \frac{x^2}{x^4 + 1} dx$$

3. Let $f(z)$ be a holomorphic function near $z = 0$ such that $f(0) + f'(0) + \cdots + f^{(n)}(0) + \cdots = \sum_{n=0}^{\infty} f^{(n)}(0)$ is convergent. Prove that
(1): $f(z)$ can be extended to a holomorphic function over $\mathbb{C}$.
(2): The series $\sum_{n=0}^{\infty} f^{(n)}(z)$ converges uniformly on $|z| \leq R$ for any $R > 0$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. (1). Let D be a domain in $\mathbb{C}$ and let f be a holomorphic function in D. Suppose that $\mathrm{Re}(f) > 0$. Prove that for any closed C^1-piecewise smooth curve C,

$$\int_C \frac{f'}{f} dz = 0.$$

(2). Use the argument principle to prove that for any $\lambda > 0$, $P(z) = z^4 + i\lambda z^3 + 1 = 0$ has exactly one solution in the first quadrant.

5. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function on the unit disc $\mathbb{D} = \{|z| < 1\} \subset \mathbb{C}$. Prove that if f is injective in the annulus $\{\frac{1}{2} < |z| < 1\}$, f must also be injective in $\mathbb{D}$.

End of Session on Complex Variables and Advanced Calculus

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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Part I. Answer all questions.

1. Let $f : [0,1] \to [0,1]$ be continuous and non-decreasing, with $f(0) = 0$ and $f(1) = 1$. For such an f, the derivative $f'(x)$ exists for Lebesgue almost every $x \in [0,1]$. Let dx denote Lebesgue measure.

a) Use Fatou's lemma to show that $\int_0^1 f'(x)dx \leqslant 1$.

b) Give an example of such a function f for which $\int_0^1 f'(x)dx < 1$, and explain why strict inequality holds.

2. Let dt denote Lebesgue measure on $\mathbb{R}$, and suppose $f \geqslant 0$ is Lebesgue integrable over $\mathbb{R}_+$, i.e.

$$\int_0^\infty f(t)dt < \infty.$$

Assume that $h > 0$.

Prove that

$$\lim_{h\to 0^+} \int_h^\infty \left(\frac{1}{h} \int_{x-h}^x f(y)dy \right) dx = \int_0^\infty f(t)dt.$$

3. Let $\{a_n\}_{n\in\mathbb{N}}$ be a real sequence satisfying $\lim_{n\to\infty} na_n = 0$. Define

$$S_n := \sum_{k=1}^n a_k$$

and

$$\sigma_n := \frac{1}{n} \sum_{k=1}^n S_k.$$

Assume that

$$\lim_{n\to\infty} \sigma_n = A.$$

Prove that

$$\lim_{n\to\infty} S_n = A.$$

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Consider a separable real Hilbert space H. (For the sake of concreteness, you may let dx be Lebesgue measure on the real line, and consider $H := L^2([0,1], dx)$, the equivalence classes of real Lebesgue square integrable functions on $[0,1]$ that differ at most on a null set, with inner product $\langle f, g\rangle := \int_{[0,1]} f(x)g(x)dx$ and norm $\|f\| := \left(\int_{[0,1]} |f(x)|^2 dx\right)^{\frac{1}{2}}$.)

a) Let $\{f_n\}_{n\in\mathbb{N}} \subset H$ be an orthonormal sequence.
Show that $\lim_{n\to\infty}\langle g, f_n\rangle = 0$ for every $g \in H$.

b) Now let $\{f_n\}_{n\in\mathbb{N}} \subset H$ be a sequence and $f \in H$ such that for every $g \in H$, one has $\lim_{n\to\infty}\langle g, (f_n - f)\rangle = 0$. Suppose that $\|f_n\| \to \|f\| \geqslant 0$ as $n \to \infty$.
Show that $\|f_n - f\| \to 0$ as $n \to \infty$.

5. Prove that if f is Lebesgue integrable on A, then, given $\varepsilon > 0$, there is $\delta > 0$ such that $\int_B |f(x)|dx < \varepsilon$ whenever $B \subseteq A$ and $|B| < \delta$.

End of Session on Real Variables and Elementary Point-Set Topology