RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2022

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Prove that the rings $\mathbb{Q}[x]/\langle x^2 - 1\rangle$ and $\mathbb{Q} \oplus \mathbb{Q}$ are isomorphic.

2. Let p be a prime. Show that any element of order p in $GL_2(\mathbb{Z}/p\mathbb{Z})$ can be conjugated to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Hint*: You may consider the p-Sylow subgroups of $GL_2(\mathbb{Z}/p\mathbb{Z})$.

3. Let a and b be elements of a field of order 2^n where n is odd. Prove that if $a^2 + ab + b^2 = 0$ then $a = b = 0$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let A, B be linear operators on a nonzero finite-dimensional vector space V over $\mathbb{C}$ such that $A^2 = B^2 = \mathrm{Id}$. Prove that there exists a nonzero subspace W of V which is invariant under A and B and $\dim W \leq 2$.

5. Let A be a complex $n \times n$ matrix. Let a_k denote the dimension of the null space of A^k (in particular, $a_0 = 0$). Prove that $a_k + a_{k+2} \leq 2a_{k+1}$ for all $k \geq 0$.

End of Session 1

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Written Qualifying Examination

Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

Problem 1. Compute $\int_{-\infty}^{\infty} \frac{\cos^2(x)}{1+x^2} dx$ using the residue theorem and contour integral.

Problem 2. Show that $2+z^2-e^{iz}$ has exactly one zero in the open upper-half plane.

Problem 3. Suppose $f(z)$ is a holomorphic function on the complex plane such that $\int\int_{\mathbf{C}} |f'(z)|^2 dxdy < \infty$. Show that f is a constant function.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

Problem 4. Let $\mathbf{D}$ be the open unit disk $\{z||z| < 1\}$ and $f : \mathbf{D} \to \mathbf{D}$ be a holomorphic function which has two fixed points. Show that f is the identity map.

Problem 5. Let Ω be an open connected set and $\{f_n : \Omega \to \mathbf{C}\}$ be a sequence of injective holomorphic functions converging uniformly on compact subsets to a non-constant holomorphic function $g : \Omega \to \mathbf{C}$. Show that g is injective.

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GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Let A be a Cantor-like set defined as follows. Given $\alpha \in (0,1)$, we remove from $A_0 = [0,1]$ an open interval $(\frac{1}{2}-\frac{\alpha}{4}, \frac{1}{2}+\frac{\alpha}{4})$ and denote by A_1 the union of the two remaining closed intervals. Next we remove the open middle intervals of length $\frac{\alpha}{2^3}$ of the two intervals constituting A_1 and denote by A_2 the union of the four remaining closed intervals. We repeat the process with each of the four intervals, removing the open middle intervals of length $\frac{\alpha}{2^5}$. Continuing the process, we obtain the sequence $(A_n)_{n\geqslant 0}$ of sets, where A_n is the union of 2^n closed intervals, and we put $A = \bigcap_{n\geqslant 0} A_n$.

(a) Show that $A \neq \varnothing$ and A is compact.

(b) Show that A is nowhere dense, i.e. $\operatorname{int}(\operatorname{cl}(A)) = \varnothing$.

(c) Show that A is perfect.

(d) Show that A is uncountable.

(e) Find the Lebesgue measure μ of A.

2. Let m denote the Lebesgue measure on $\mathbb{R}$. Suppose that f is nonnegative and measurable function on a set $A \subseteq \mathbb{R}$ such that $m(A) < \infty$. Prove that $f \in L^1(A)$ if and only if

$$\sum_{k=0}^{\infty} k m(A_k) < \infty,$$

where $A_k = \{x \in A : k \leqslant f(x) < k+1\}$.

3. Let m denote the Lebesgue measure on $\mathbb{R}$. Let $I = [0,1]$. Decide whether the sets A and B below are closed in $L^1(I,m)$, where

$$A := \left\{ f \in L^1(I,m) : \int_I |f(x)|^2 dm \geqslant 1 \right\}$$

and

$$B := \left\{ f \in L^1(I,m) : \int_I |f(x)|^2 dm \leqslant 1 \right\}$$

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $(X, \mathcal{B}(X), \mu)$ be a finite measurable space. Assume that two sequences $\{f_n\}$ and $\{g_n\}$ of measurable functions satisfy $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ (converge measure). Show that the product $f_n g_n \xrightarrow{\mu} fg$ converge in the measure. Is the statement true is $\mu(X)$ is infinite?

5. Let $\Lambda(\mathbb{R}^3)$ be the Lebesgue σ-algebra of $\mathbb{R}^3$, and let λ denote the three-dimensional Lebesgue measure on $\Lambda(\mathbb{R}^3)$. Let $\mathcal{P}(\mathbb{R}^3)$ denote the power set of $\mathbb{R}^3$. Clearly $\Lambda(\mathbb{R}^3) \subset \mathcal{P}(\mathbb{R}^3)$. Show that λ cannot be extended to $\mathcal{P}(\mathbb{R}^3)$.

Hint: You may use the Banach-Tarski paradox, which states the following:
Theorem *If U and V are arbitrary bounded open sets in $\mathbb{R}^n$, with $n \geqslant 3$, then there exists a $k \in \mathbb{N}$ and subsets $E_1, ..., E_k, F_1, ..., F_k$ of $\mathbb{R}^n$ such that*

(i) $E_j \cap E_l = \varnothing$ *whenever* $j \neq l$, *and* $\bigcup_j E_j = U$;
(ii) $F_j \cap F_l = \varnothing$ *whenever* $j \neq l$, *and* $\bigcup_j F_j = V$;
(iii) $E_j \sim F_j$ *for all* $j = 1, ..., k$.

Here $\sim$ means Euclidean congruence, i.e. $A \sim B$ if A can be mapped into B by a combination of a translation, a rotation, and a reflection.

End of Session on Real Variables and Elementary Point-Set Topology