RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer all of the questions in Part I (numbered 1, 2, 3).

Answer one of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state clearly which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. Only material in the examination book(s) will be graded, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

• Be sure your special exam ID code symbol is on each exam book that you are submitting.

• Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.

• Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.

• At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.
Part I. Answer all questions.

1. Let $G$ be a group and $Z(G)$ the center of $G$. Show that the group $G/Z(G)$ does not have prime order. Find a group $G$ such that $G/Z(G)$ has 4 elements.

2. Show that every prime ideal $P$ in $\mathbb{Z}[x]$ which is not principal contains a prime number.

3. Show that every finite noncyclic group is a finite union of proper subgroups, and that if a group maps surjectively to a finite noncyclic group then it is a finite union of proper subgroups and use this to determine for which positive integers the product of $n$ copies of the integers is a finite union of proper subgroups.
Part II. Answer one of the two questions. If you work on both questions, indicate clearly which one should be graded.

4. Let $A$ and $B$ be two square matrices over a field $F$. Suppose $\text{diag}(A, A)$ and $\text{diag}(B, B)$ are similar. Show that $A$ and $B$ are similar.

5. (A) Suppose that $p$ and $q$ are distinct primes and a group $G$ is generated by elements of order $p$ and also by elements of order $q$. Show that any homomorphism of $G$ to an abelian group is trivial.

(B) Show that for $n \geq 5$ the alternating group $A_n$ of even permutations of $n$ objects is generated by elements of order 2, and also by elements of order 3, so that for such $n$ the only homomorphisms to abelian groups are trivial.

End of Session on Algebra