RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination
August, 2021

Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.

- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.

- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.

- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.
Part I. Answer all questions.

1. (a) For a sequence \( \{a_n\}_{n=1}^\infty \) of real numbers, write down the definition of

\[
\limsup_{n \to \infty} a_n.
\]

(b) Prove that, for any sequence of Lebesgue measurable functions \( f_n : \mathbb{R} \to [0, 1], \ n = 1, 2, \ldots \), the function

\[
f = \limsup_{n \to \infty} f_n
\]

is also Lebesgue measurable.

2. Consider the \( \ell^1 \) space

\[
\ell^1 = \{ x = (x_1, x_2, \ldots) : x_k \in \mathbb{R}, \|x\|_1 < \infty \}, \quad \|x\|_1 = \sum_{k=1}^\infty |x_k|,
\]

and endow the space with the metric \( \rho(x, y) = \|x - y\|_1 \). Prove that

\[
K = \{ x = (x_1, x_2, \ldots) : |x_k| \leq k^{-2}, k = 1, 2, \ldots \}
\]

is a compact set in \( \ell^1 \).

3. We say a function \( g : [0, 1] \to \mathbb{R} \) is Lipschitz if

\[
\sup \left\{ \frac{|g(x) - g(y)|}{|x - y|} : x, y \in [0, 1], x \neq y \right\} < \infty.
\]

Prove that, for any non-decreasing and absolutely continuous function \( f : [0, 1] \to \mathbb{R} \), there exists a sequence of functions \( f_n : [0, 1] \to \mathbb{R}, \ n = 1, 2, \ldots \), such that each \( f_n \) is non-decreasing, Lipschitz, and

\[
f_1 \leq f_2 \leq \ldots \quad \text{and} \quad \lim_{n \to \infty} f_n = f \quad \text{everywhere on} \ [0, 1].\]
Part II. Answer one of the two questions.  
If you work on both questions, indicate clearly which one should be graded.

4. Let \( f_n : [0, \infty) \to \mathbb{R}, n = 1, 2, \ldots, \) be such that

\[
\lim_{n \to \infty} f_n(x) = e^{-x} \text{ for every } x \in [0, \infty),
\]

\[
\sup_n \int_0^\infty |f_n(x)|^2 e^{-x} \, dx < \infty,
\]

\[
\limsup_{n \to \infty} \int_0^\infty |f_n(x)| \, dx \leq 1.
\]

Prove that

\[
\lim_{n \to \infty} \int_0^\infty |f_n(x) - e^{-x}| \, dx = 0.
\]

5. Let \((\Omega, \mathcal{A}, \mu)\) be a measure space and \(f : \Omega \to [0, 1]\) be \(\mathcal{A}\)-measurable.

(a) Let \(\mathcal{B}_{[0,1]}\) denote the Borel \(\sigma\) algebra on \([0,1]\). Prove that the set

\[
H = \{(x, y) : x \in \Omega, y \in [0, f(x)]\} \subset \Omega \times [0, 1]
\]

is \((\mathcal{A} \times \mathcal{B}_{[0,1]})\)-measurable.

(b) Assume further that \(\mu(\Omega) < \infty\), and let \(m\) denote the Lebesgue measure on \([0, 1]\). Prove that

\[
(\mu \times m)(H) = \int_{\Omega} f \, d\mu.
\]