RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2021

Session on Algebra

The Qualifying Examination consists of three two-hour sessions. This is the session on Algebra. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. The following are four classes of commutative rings, in alphabetical order

   - fields;
   - integral domains (IDs);
   - principal ideals domains (PIDs);
   - unique factorization domains (UFDs).

   These are contained in one-another, in some order, so that

   $$A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq A_4.$$

   (a) Determine the order;

   (b) Give an example in each class to show that the inclusions are proper.

2. (a) If R is a commutative ring, define what it means for R to be Noetherian and state Hilbert's basis theorem.

   (b) Give an example of a non-Noetherian commutative ring.

3. Let G be a group of order 105 and let P_3, P_5, and P_7 be Sylow 3, 5, and 7 subgroups, respectively. Assuming the Sylow theorems, prove the following:

   (a) At least one of P_5 or P_7 is normal in G;

   (b) G has a cyclic subgroup of order 35;

   (c) Both P_5 and P_7 are normal in G.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

1. Find all similarity classes of 2×2 matrices A with entries in $\mathbb{Q}$ satisfying $A^4 = I$. What are the corresponding rational canonical forms?

2. (a) Find the possible Jordan Canonical Forms of any matrix such that $A^4 = I$ over $F = \mathbb{F}_5$.

   (b) Give an example of a matrix B over $F = \mathbb{F}_3$ satisfying $B^4 = I$, such that B is not diagonalizable.

End of Session on Algebra

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Session on Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the session on Complex Variables and Advanced Calculus. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. Prove that all five roots of $2z^5 + 8z - 1$ lie in the disc $|z| < 2$ but only one root lies inside $|z| < 1$. [Hint: Apply Rouché's Theorem twice.]

2. Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function which satisfies:

$$|f(z)| \leqslant 1 \text{ and } f(i) = 0.$$

Prove that for all $z \in \mathbb{H}$,

$$|f(z)| \leqslant \left| \frac{z-i}{z+i} \right|.$$

(Here $\mathbb{H}$ is the upper half-plane, $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$.)

3. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 - (\frac{\pi}{2})^4} dx$$

using Cauchy's calculus of residues. [Hint: does it even converge near $x = \pm\pi/2$?]

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic and suppose there is an open set U whose closure $\overline{U} \subset \mathbb{D}$ is in the disk, such that f is injective on U. Must there exist an open set W with $\overline{U} \subset W \subset \mathbb{D}$ such that f is injective on W? If so, prove your answer, and if not, provide a counterexample. (Here $\mathbb{D}$ is the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.)

5. Let R be the parallelogram with vertices $(0,0)$, $(1,1)$, $(3,0)$, and $(2,-1)$. Evaluate the integral

$$\iint_R (x+2y)^2 e^{(x-y)} dA.$$

End of Session on Complex Variables and Advanced Calculus

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2021

Session on Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the session on Real Variables and Elementary Point-Set Topology. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
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- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Prove that a metric space (X, ρ) is complete if and only if for every decreasing sequence $F_1 \supseteq F_2 \supseteq \dots$ of nonempty closed subsets of X with $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$, the intersection $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$ for some $x_0 \in X$. *Hint:* For any set $E \subseteq X$ we define its diameter by setting $\operatorname{diam}(E) = \sup\{\rho(x, y) : x, y \in E\}$.

2. Recall that a subset E of a metric space X is nowhere dense if $\operatorname{int}(\operatorname{cl}(E)) = \varnothing$.

(A) Is it true that every nowhere dense subset of $\mathbb{R}$ must have Lebesgue measure zero? Justify your answer.

(B) Give an example of a nowhere dense and uncountable subset of $\mathbb{R}$ which has Lebesgue measure zero.

(C) Is it true that every subset of the standard Cantor set $\mathcal{C}$ in $[0, 1]$ is Lebesgue measurable? Justify your answer.

3. Let $(X, \mathcal{B}(X), \mu)$ and $(Y, \mathcal{B}(Y), \nu)$ be two σ-finite measure spaces and let $1 \leqslant p < \infty$. Show that for every nonnegative measurable function F on the product space $X \times Y$ with the product measure $\mu \times \nu$ we have

$$\left[\int_Y \left(\int_X F(x,y)d\mu(x)\right)^p d\nu(y)\right]^{1/p} \leqslant \int_X \left[\int_Y F(x,y)^p d\nu(y)\right]^{1/p} d\mu(x).$$

Hint: Observe that

$$\left(\int_X F(x,y)d\mu(x)\right)^p = \left(\int_X F(x,y)d\mu(x)\right)\left(\int_X F(x,y)d\mu(x)\right)^{p-1}$$

and apply Hölder's inequality.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $(X, \mathcal{B}(X), \mu)$ be a measure space. If $f_n, g_n, f, g \in L^1(X, \mu)$, and

(i) $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ a.e.;

(ii) $|f_n| \leqslant g_n$ for all $n \in \mathbb{N}$;

(iii) $\lim_{n\to\infty} \int_X g_n(x) d\mu(x) = \int_X g(x) d\mu(x)$. Then one has

$$\lim_{n\to\infty} \int_X f_n(x) d\mu(x) = \int_X f(x) d\mu(x).$$

5. Assume that $E \subseteq \mathbb{R}$ is Lebesgue measurable and $0 < m(E) < \infty$.

(A) Show that if E is bounded and $m(E) = p > 0$, then for each $q \in (0, p)$ there is a measurable set $B \subseteq E$ of measure q.

(B) Prove that for any $0 < \alpha < 1$ there is an open interval I such that

$$\alpha m(I) \leqslant m(E \cap I).$$

End of Session on Real Variables and Elementary Point-Set Topology