Session 3. Real Variables and Elementary Point-Set Topology Written Qualifying Exam

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer all of the questions in Part I (numbered 1, 2, 3).

Answer one of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state clearly which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. Only material in the examination book(s) will be graded, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

• Be sure your special exam ID code symbol is on each exam book that you are submitting.

• Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.

• Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.

• At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.
Part I. Answer all questions.

1. Let $E_k$ be a sequence of Lebesgue measurable subsets of $\mathbb{R}$. Let 
   
   \[ E = \{ x \in \mathbb{R} : x \in E_k \text{ for infinitely many } k \} . \]

   1. Show that $E$ is Lebesgue measurable.

   2. Show that if $\sum_{i=1}^{\infty} |E_k| < \infty$, then $|E| = 0$.

   3. Assume instead only that $\lim_{k \to \infty} |E_k| = 0$. Must $|E| = 0$?

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of integrable functions with $\int_{\mathbb{R}} |f_n| \leq 1$. Assume that there exists a measurable function $f$ such that $|\{ x \in \mathbb{R} : |f_n(x) - f(x)| > \epsilon \}| \to 0$ as $n \to \infty$ for each $\epsilon > 0$.

   1. Show that there is a subsequence $f_{n_k}$ which converges to $f$ almost everywhere.

   2. Show that $f$ is integrable.

3. Let $M$ be a metric space, $N$ be a complete metric space, and $S \subset M$ is a dense subset. Let $f : S \to N$ be a uniformly continuous function. Show that there exists a unique continuous function $g : M \to N$ such that $g|_S = f$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Construct a nondecreasing function $f : (0, 1) \to \mathbb{R}$ whose discontinuity set is exactly $\mathbb{Q} \cap (0, 1)$ (the rational numbers in $(0, 1)$), or prove that such a function does not exist.

5. Let $f(x, y) = \frac{xy}{(\frac{x}{|x|} + |y|)^\alpha}$ for $(x, y) \neq 0$, and $f(0, 0) = 0$, where $\alpha \in \mathbb{R}$. For what values of $\alpha$ is $f$ integrable on $(-1, 1) \times (-1, 1)$? Justify your answer.

End of Session 1