RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2020

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as “Book 1 of X”, “Book 2 of X”, etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don’t want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Suppose that A is a not necessarily commutative, finite dimensional associative algebra with a unit over a field F and $P \lhd A$ is a two-sided ideal such that for $a, b \in A$, $ab \in P$ implies $a \in P$ or $b \in P$. Show that A/P must be a division algebra (i.e. every nonzero element has a multiplicative inverse).

2. Show that every group of order 2020 contains a unique (and hence normal) subgroup of order 505.

3. Let M be a matrix with integer entries.

(a) Prove that the minimal polynomial of M over $\mathbb{C}$

$$f_{min}(t) = t^k + \sum_{i=0}^{k-1} a_i t^i$$

has integer coefficients.

(b) Prove that if M is diagonalizable over $\mathbb{Q}$ then there exists an integer N such that the matrix M mod p is diagonalizable over $\mathbb{Z}/p\mathbb{Z}$ for all $p > N$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let F be a field and let L be the ring of Laurent polynomials $L = F[x, x^{-1}]$ (it is the subring of $F(x)$ generated over F by x and x^{-1}). We consider L as a module over the ring of polynomials $R = F[x]$.

(a) Show that L is not a finitely generated module over R.

(b) Show that every finitely generated submodule of L is free with a single generator.

5. Let R be a commutative integral domain and let $I \lhd R$ be an ideal.

(a) Show that every alternating bilinear form

$$f : I \times I \to R$$

is zero.

(b) Show that if R is a principal ideal domain, then every alternating bilinear form

$$f : I \times I \to M$$

to any R-module M is zero.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Assume that for any $z \in D_R(a) = \{w \in \mathbb{C} : |w - a| < R\}$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

(A) If $0 \leqslant r < R$, prove that

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{i\theta})|^2 d\theta.$$

(B) Using the identity from part **(A)** deduce that every bounded entire function is constant.

2. Use the calculus of the residues to show that

$$\int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}.$$

3. Let f be holomorphic on a neighborhood of the closed unit disc centered at the origin. Assume that $|f(z)| = 1$ if $|z| = 1$, and f is non-constant. Prove that the image of f contains the unit disc.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Assume that z_1, z_2, w_1, w_2 are points in the upper half plane

$$H^+ := \{z \in \mathbb{C} : \ \text{Im} z > 0\}.$$

Does there always exist a one to one and onto holomorphic self-map ϕ of H^+ such that $\phi(z_1) = w_1$ and $\phi(z_2) = w_2$? If not, give a necessary and sufficient condition for such a map ϕ to exist.

5. Let $f(z)$ be a holomorphic function in a neighborhood of $z = 0$ which satisfies

$$f(z) = z + f(z^2).$$

(A) Find the (unique up to additive constant) power series of f and prove that f can be analytically continued to $|z| < 1$.

(B) Prove that f can not be analytically continued to any connected open set that contains $|z| < 1$ and is strictly larger than it.

End of Session 2

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GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

January 2020

Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. (a) Exhibit a sequence of functions $f_n : [0,1] \to [0,1]$ so that $f_n \to 0$ in L^1 but for every $x \in [0,1]$, $\{f_n(x)\}$ has no limit.

(b) Must there exist a subsequence of f_n converging pointwise?

2. State and prove Fatou's Lemma.

3. Suppose f is defined on $\mathbb{R}^2$ as follows: $f(x,y) = a_n$ if $n \leqslant x < n+1$ and $n \leqslant y < n+1$, $(n \geqslant 0)$; $f(x,y) = -a_n$ if $n \leqslant x < n+1$ and $n+1 \leqslant y < n+2$, $(n \geqslant 0)$; while $f(x,y) = 0$ elsewhere. Here $a_n = \sum_{k \leqslant n} b_k$, with $\{b_k\}$ a positive sequence such that $\sum_{k=0}^{\infty} b_k = s < \infty$.

(a) Verify that the slice $f_x(y) = f(x,y)$ is integrable, and show that for all x, $\int f_x(y)dy = 0$. Hence $\int \left(\int f(x,y)dy\right) dx = 0$.

(b) Prove that the slice $f^y(x) = f(x,y)$ is integrable and that $\int f^y(x)dx = a_0$ if $0 \leqslant y < 1$, and $\int f^y(x)dx = a_n - a_{n-1}$ if $n \leqslant y < n+1$ with $n \geqslant 1$. Hence the function $y \mapsto \int f^y(x)dx$ is integrable on $(0,\infty)$ and $\int \left(\int f(x,y)dx\right) dy = s$.

(c) Prove directly that $\int_{\mathbb{R}\times\mathbb{R}} |f(x,y)|dxdy = \infty$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let f be absolutely continuous on $[a, b]$. Is it true that the total variation function $TV(f_{[a,x]})$ is absolutely continuous? If so prove it, if not, provide a counterexample.

5. Let $\tau(0) = \infty$, and for a non-zero integer $c \in \mathbb{Z}$, let $\tau(c)$ denote the smallest positive integer which does not divide c; so $\tau(12) = 5$ but $\tau(13) = 2$. Define the function $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_{\geqslant 0}$ by $d(a, b) = 2^{-\tau(a-b)}$ (of course $2^{-\infty} = 0$).

(a) Prove that $d(\cdot, \cdot)$ is a metric on the integers.

(b) Describe explicitly the distance 2^{-n} ball about a point $a \in \mathbb{Z}$, that is

$$\{b \in \mathbb{Z} : d(a, b) < 2^{-n}\} = \,?$$

[Hint: Let N_n denote the least common multiple of the numbers $1, \ldots, n$.]

(c) From your answer in part (b), prove that the topology on $\mathbb{Z}$ induced by this metric is generated by the set of all non-constant arithmetic progressions, $q\mathbb{Z} + a = \{qm + a\}_{m \in \mathbb{Z}}$, with $q \geqslant 1$.

(d) Prove that the complement of an arithmetic progression is open (so arithmetic progressions are both open and closed).

(e) Compute the complement of

$$\bigcup_{p \text{ prime}} (p\mathbb{Z} + 0),$$

where the union runs over primes.

(f) Conclude from (e) that there are infinitely many primes. [Hint: what is the cardinality of a non-empty open set?]

End of Session 3