This examination booklet contains 6 questions on 10 pages of paper including the front cover. 
Do all of your work in this booklet, show all your computations and justify/explain your answers. 
Do not remove any pages. 
Your justification must be based on techniques already discussed in this course. If asked to evaluate an integral, remember to show all the steps that gives you its value. 
Except for your personal note sheet, no other resources like class notes, books, calculator, etc are allowed. Remember that your note sheet must be handwritten, on both sides of a single sheet of paper. 
Unless otherwise stated, give exact answers. For example, write $\pi$ and $\sqrt{2}$ instead of 3.14 and 1.41. However, when an expression simplifies to a well-known value, you must use that value. For example, you must write 1 instead of $e^0$, and you must write $\pi/3$ instead of $\sec^{-1}(2)$. 
If you run out of space when answering a problem, you may use any of the last three pages of the exam, but you must: indicate in the space below the question that you are continuing your answer on the extra sheet, and indicate on the extra sheet which problem you are working on. 
If asked to use a specific theorem to calculate some quantity, you will receive no points if that theorem is not used. 
Do not discuss the exam with anyone until grades are posted on Canvas.
Problem 1. [* points] Suppose \( \mathbf{i} = <1, 0> \) and \( \mathbf{j} = <0, 1> \) are the standard unit vectors in \( \mathbb{R}^2 \).

a) Let \( \mathbf{v} \) represent the unit vector in the direction of the vector \( \mathbf{i} + \mathbf{j} = <1, 1> \). Then \( \mathbf{v} = \)

b) Suppose \( \mathbf{u} \) is the vector \( \mathbf{u} = <x, y> \). Find the values of \( x \) and \( y \) if \( \text{proj}_\mathbf{v} \mathbf{u} = 9 \mathbf{v} \), \( \text{proj}_\mathbf{j} \mathbf{u} = 4 \mathbf{j} \)

\[ \text{sol} \]

a) \( \mathbf{v} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \)

b) If \( \text{proj}_\mathbf{j} \mathbf{u} = 4 \mathbf{j} \), then \( \frac{x \mathbf{j} + y \mathbf{j}}{|\mathbf{j}|^2} \mathbf{j} = 4 \mathbf{j} \), or \( y \mathbf{j} = 4 \mathbf{j} \), which implies \( y = 4 \).

If \( \text{proj}_\mathbf{v} \mathbf{u} = 9 \mathbf{v} \), then \( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = 9 \mathbf{v} \), or \( \mathbf{u} \cdot \mathbf{v} = 9 \), or \( \frac{x + y}{\sqrt{2}} = 9 \), which gives \( x + y = 9\sqrt{2} \), or \( x = 9\sqrt{2} - 4 \).
Problem 2. [* points]
a) Find a parametrization of the curve \( r(t) \) obtained as the intersection of the sphere \( x^2 + y^2 + z^2 = 49 \) with the plane \( z = 3y \).
b) Find the arc-length parameter \( s(t) \) as a function of \( t \).
c) Find the length of the curve.

**sol**

a) substitute the plane equation into the sphere to get \( x^2 + 10y^2 = 49 \). Call \( Y = \sqrt{10}y \). Notice that this gives \( x^2 + Y^2 = 49 \) so \( x = 7\cos t \) and \( Y = 3\sin t \), that is \( y = \frac{7}{\sqrt{10}} \sin t \). Therefore, the parametrization of the curve is

\[
\mathbf{r}(t) = \left(7\cos t, \frac{7\sin t}{\sqrt{10}}, \frac{21\sin t}{\sqrt{10}}\right)
\]

b) The velocity is

\[
\mathbf{v}(t) = 7\left(-\sin t, \frac{\cos t}{\sqrt{10}}, \frac{3\cos t}{\sqrt{10}}\right)
\]

and the speed is

\[
|\mathbf{v}| = 7\sqrt{\sin^2 t + \frac{\cos^2 t}{10} + \frac{9\cos^2 t}{10}} = 7
\]

Therefore, the arc-length parameter function is

\[
s(t) = \int_0^t |\mathbf{v}| \, du = 7t
\]

and the length is \( s(2\pi) = 14\pi \).
Problem 3. [* points] Consider the planes with equations $a_1 x + y - 4z = 2$ and $a_2 x + y + z = 10$, where $a_1, a_2$ are constants.

a) What must be the values of $a_1$ and $a_2$ so that the line of intersection between these two planes has a direction vector $v = \langle 5, 0, -5 \rangle$?

b) For the values of $a_1$ and $a_2$ from part a), if $P = (X, Y, 0)$ is a point in this line of intersection, what are the values of $X$ and $Y$?

c) Write the parametric equations for this line so that at time $t = 0$, it passes through the point $P$.

sol
a) The normal vectors are $n_1 = (a_1, 1, -4)$ and $n_2 = (a_2, 1, 1)$. The cross product of these vectors will give a direction for this line: since $n_1 \times n_2 = \langle 5, -a_1 - 4a_2, a_1 - a_2 \rangle$ and comparing with $v = (5, 0, -5)$, we see that $a_1 + 4a_2 = 0$, $a_1 - a_2 = -5$. Thus $a_2 = 1$ and $a_1 = -4$.

Alternatively, you can subtract the plane equations to obtain $(a_2 x + y + z) - (a_1 x + y - 4z) = 10 - 2$ or $(a_2 - a_1)x + 5z = 8$ or $z = \frac{8}{5} - \frac{(a_2 - a_1)}{5} x$. Replacing this in the second equation we get $y = 10 - a_2 x - z = 10 - a_2 x - \frac{8}{5} + \frac{(a_2 - a_1)}{5} x = \frac{42}{5} + \frac{(-4a_2 - a_1)}{5} x$ so the parametric equations of the line are $x = t$, $y = \frac{42}{5} - \frac{(a_1 + 4a_2)}{5} t$, $z = \frac{8}{5} - \frac{(a_2 - a_1)}{5} t$. The direction vector is $\langle 1, -\frac{a_1 + 4a_2}{5}, -\frac{(a_2 - a_1)}{5} \rangle$, thus comparing with $\langle 5, 0, -5 \rangle = 5 \langle 1, 0, -1 \rangle$, we see that we should have $a_1 + 4a_2 = 0$ and $a_2 - a_1 = 5$, which gives the same equations as before.

b) we must solve $-4X + Y = 2$ and $X + Y = 10$. We obtain $X = \frac{8}{5}$ and $Y = \frac{42}{5}$

c) $x = \frac{8}{5} + 5t$, $y = \frac{42}{5}$, $z = -5t$
Problem 4. [* points] Determine whether the lines with parametric equations \( x = 2 - t, y = 3t, z = 1 + t \) and \( x = 5 + 2s, y = 1 - s, z = 8 + 3s \) intersect or not. If they intersect, find the point of intersection.

sol:
We must solve
\[
\begin{aligned}
2 - t &= 5 + 2s \\
3t &= 1 - s \\
1 + t &= 8 + 3s \\
\end{aligned}
\]
The last equation says that \( t = 7 + 3s \), and substituting in the first we get
\[
2 - 7 - 3s = 5 + 2s \implies s = -2
\]
Substituting in the second equation we get
\[
21 + 9s = 1 - s \implies s = -2
\]
Thus, the lines intersect at the point \((1, 3, 2)\).
Problem 5. [* points] a) What is the domain of the function

\[ f(x, y) = \frac{xy\sqrt{x + y}}{x^2 + y^2} \]

Sketch your answer. b) Determine whether

\[ \lim_{(x,y) \to (0,0)} \frac{xy\sqrt{x + y}}{x^2 + y^2} \]

exists or not. Be sure to fully justify your answer either way.

sol

a) We need \( x^2 + y^2 \neq 0 \) since the denominator cannot become zero and also \( x + y \geq 0 \). The second becomes \( y \geq -x \) which is the region of the plane above this line (first quadrant and “half” of the second and fourth quadrants).

b) We switch to polar coordinates

\[ \lim_{r \to 0} \frac{r^2 \cos \theta \sin \theta \sqrt{r \cos \theta + r \sin \theta}}{r^2} = \lim_{r \to 0} \sqrt{r} \cos \theta \sin \theta \sqrt{\cos \theta + \sin \theta} \]

Notice that \(-2 \leq \cos \theta \sin \theta \sqrt{\cos \theta + \sin \theta} \leq 2\) thus

\[ -2\sqrt{r} \leq \sqrt{r} \cos \theta \sin \theta \sqrt{\cos \theta + \sin \theta} \leq 2\sqrt{r} \]

so by the squeeze theorem, since the limit of the left and right sides is zero as \( r \) approaches zero, the limit for the middle one is zero as well.
Problem 6. [* points] The following two questions are independent of one another.

a) Suppose that the level curve of a function \( f(x, y) \) consists of straight lines. Must the graph \( z = f(x, y) \) correspond to a plane?

b) For the function \( f(x, y) = x^2 - y \), find the equation of the level curve passing through the point \( P = (2, 1) \).

sol:

a) no. For example, consider \( f(x, y) = \sqrt{x + y} \). The level curves will be straight lines, but this is not the graph of a plane.

b) \( f(2, 1) = 4 - 1 = 3 \), so the level curve equation is \( x^2 - y = 3 \).
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