This examination booklet contains 6 questions on 10 pages of paper including the front cover.

- Do all of your work in this booklet, show all your computations and justify/explain your answers. Your justification must be based on techniques already discussed in this course. If asked to evaluate an integral, remember to show all the steps that gives you its value.

- Except for your personal note sheet, no other resources like class notes, books, calculator, etc are allowed. Remember that your note sheet must be handwritten, on both sides of a single sheet of paper.

- Unless otherwise state, give exact answers. For example, write \( \pi \) and \( \sqrt{2} \) instead of 3.14 and 1.41. However, when an expression simplifies to a well-known value, you must use that value. For example, you must write 1 instead of \( e^0 \), and you must write \( \pi/3 \) instead of \( \sec^{-1}(2) \).

- If you run out of space when answering a problem, you may use any of the last three pages of the exam, but you must: indicate in the space below the question that you are continuing your answer on the extra sheet, and indicate on the extra sheet which problem you are working on.

- Do not discuss the exam with anyone until grades are posted on Canvas.

This section is reserved for the instructor or grader

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WRITE OUT AND SIGN PLEDGE [1 point]

On my honor, I have neither received nor given any unauthorized assistance on this examination.
Problem 1. [16 points] a) Find a parametrization of the curve \( r(t) \) obtained as the intersection of the sphere \( x^2 + y^2 + z^2 = 49 \) with the plane \( z = 3y \).

\[
\begin{align*}
x^2 + 9y^2 & = 49 \\
x^2 + 10y^2 & = 49
\end{align*}
\]

\[
x = 7 \cos t, \quad y = \frac{7}{\sqrt{10}} \sin t, \quad z = \frac{21}{\sqrt{10}} \sin t
\]

b) Find the arc-length parameter \( s(t) \) as a function of \( t \).

\[
\left( \frac{dr}{dt} \cdot \frac{dr}{dt} \right)^{\frac{1}{2}} = 7 \sqrt{\sin^2 t + \frac{\cos^2 t}{10} + \frac{9\cos^2 t}{10}} = 7
\]

so the speed is
\[
s(t) = \int_0^t \left( \frac{dr}{dt} \cdot \frac{dr}{dt} \right)^{\frac{1}{2}} dt = 7t
\]

c) Find the length of the curve.

\[
\text{Length} = \int_0^{2\pi} |\text{dr}| dt = 14\pi
\]

d) Identify the previous curve, for example, as a circle, straight line, helix, etc.

It is a circle
Problem 2. [14 points] Suppose that \( T(x, y) \) represents the temperature at the point \((x, y)\). Assume that at the origin \((0,0)\), \( T \) is increasing at a rate of 3 F/mile, in the direction of the vector \( \mathbf{j} \) and it is decreasing at a rate of \( \sqrt{3} \) F/mile in the direction of the vector \( -\mathbf{i} \).

a) What is the gradient of \( T(x, y) \) when evaluated at the point \((0,0)\)?

\[
\nabla T(0,0) \cdot \mathbf{j} = \frac{3 \text{ F}}{\text{mile}}
\]

\[
\nabla T(0,0) \cdot (-\mathbf{i}) = \frac{-\sqrt{3} \text{ F}}{\text{mile}}
\]

\[
\frac{\partial T}{\partial y}(0,0) = \frac{3 \text{ F}}{\text{mile}}
\]

\[
\frac{\partial T}{\partial x}(0,0) = \frac{-\sqrt{3} \text{ F}}{\text{mile}}
\]

\[
\nabla T(0,0) = \left( \frac{\sqrt{3} \mathbf{i}}{3}, \frac{-\sqrt{3}}{3} \mathbf{j} \right) \frac{\text{F}}{\text{mile}}
\]

b) Write a unit direction vector \( \mathbf{v} \) with negative first coordinate in which the rate of the temperature at the origin \((0,0)\) is zero.

\[
\mathbf{v} = (v_1, v_2)
\]

\[
\nabla T \cdot \mathbf{v} = 0
\]

\[
(\sqrt{3}, 3) \cdot (v_1, v_2) = 0
\]

\[
\sqrt{3}v_1 + 3v_2 = 0
\]

\[
v_2 = -\frac{\sqrt{3}}{3}v_1
\]

Also, \( |\mathbf{v}| = \sqrt{v_1^2 + v_2^2} = 1 \)

\( v_1^2 + v_2^2 = 1 \)

\( v_1^2 + \frac{1}{3}v_1^2 = 1 \)

\( \frac{4}{3}v_1^2 = 3 \)

\( v_1 = \pm \frac{\sqrt{3}}{2} \)

\( v_1 = -\frac{\sqrt{3}}{2} \)

\( v_2 = (-\frac{\sqrt{3}}{3})(-\frac{\sqrt{3}}{2}) = \frac{1}{2} \)

\[
\mathbf{v} = \left( -\frac{\sqrt{3}}{2}, \frac{1}{2} \right)
\]
Problem 3. [15 points] Suppose that \( f(x, y) \) is a function of \( x \) and \( y \), and \( x = 4s + t \), \( y = s - 2t \). Then

\[
\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}
\]

\[
= 4 \frac{\partial f}{\partial x} + 2 \frac{\partial f}{\partial y}
\]

b) \( \frac{\partial f}{\partial t} = \)

\[
\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}
\]

\[
= -2 \frac{\partial f}{\partial x} - 2 \frac{\partial f}{\partial y}
\]

Thus \( \frac{\partial f}{\partial s} \frac{\partial f}{\partial t} = \left( 4 \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \right) \left( \frac{\partial f}{\partial x} - 2 \frac{\partial f}{\partial y} \right) \)

c) Now suppose that \( f(x, y) = e^{Ax+By} \) where \( A, B \) are non-zero constants. Define \( g(s, t) = f(x(s, t), y(s, t)) \) where \( x(s, t) = 4s + t \), \( y(s, t) = s - 2t \) is the change of variables introduced at the beginning of the problem. What must be the ratio \( \frac{B}{A} \) if we want the directional derivative of \( g(s, t) \) in the direction of the vector \( u = 2i \) to be zero.

We want \( D_{u} g \) to be zero,

but \( D_{u} g = 2g_{s} \) so we need

\[
\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x} \cdot 4 \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \leq 0
\]

\[4A e^{Ax+By} + B e^{Ax+By} = 0\]

\[B = -4A \rightarrow A = -4\]
Problem 4. [15 points] Suppose one is interested in studying \( \lim_{(x,y) \to (0,0)} \frac{x^2y}{x^4+y^2} \).

a) What is the previous limit if one approaches \((0,0)\) using the path \( y = mx \), where \( m \) is a non-zero number.

\[
\frac{xy}{x^4+y^2} = \frac{x^2(mx)}{x^4+7m^2x^2} = \frac{mx}{x^2+7m^2} \\
\text{so the limit along this path is} \\
\lim_{x \to 0} \left( \frac{mx}{x^2+7m^2} \right) = 0, \quad \quad m \neq 0
\]

b) Suppose one approaches \((0,0)\) using the path \( y = bx^2 \). What are the values of \( b \) for which the limit along this path gives \( \frac{1}{8} \).

\[
\frac{x^2y}{x^4+y^2} = \frac{y=x^2}{x^4+7b^2x^4} = \frac{b}{1+7b^2} \\
\text{so the limit along this path is} \\
\lim_{x \to 0} \frac{b}{1+7b^2} = \frac{b}{1+7b^2} \\
\text{we want this to equal} \frac{1}{8}, \text{ so} \\
\frac{b}{1+7b^2} = \frac{1}{8} \\
\text{or} \\
8b = 1+7b^2 \\
0 = 7b^2-8b+1 \\
0 = (b-1)(7b-1) \\
\left\{ b = 1, \quad \frac{1}{7} \right\}
\]

c) What can you conclude about the existence or not of \( \lim_{(x,y) \to (0,0)} \frac{x^2y}{x^4+y^2} \).

\text{Does not exist}
Problem 5. [16 points] Let \( f(x, y, z) = \ln(x^ay^bz) \), \( P = (1, 1, 1) \). Here \( a \) and \( b \) are non-zero integers.

a) Find the equation of the gradient at the point \( P \).

\[
 \nabla f = \left( \frac{a x^{a-1} y^b z}{x^a y^b z}, \frac{b x^a y^{b-1} z}{x^a y^b z}, \frac{x^a y^b z}{x^a y^b z} \right)
\]

\[
 \nabla f(P) = (a, b, 1)
\]

b) If \( b = 6 \), find the possible values of \( a \) if it is known that the largest rate of increase of \( f \) at the point \( P \) equals \( \sqrt{46} \).

The largest rate of increase

\[
= |\nabla f| = \sqrt{a^2 + b^2 + 1}
\]

\[
= \sqrt{a^2 + 36 + 1} = \sqrt{46}
\]

\[ a^2 + 37 = 46 \]

\[ a^2 = 9 \]

\[ a = \pm 3 \]
Problem 6. [17 points] The equation $x^2 + y^2 + z^2 = z^3$ defines a surface. Find the point(s) $P$ on this surface where the tangent plane to the surface at $P$ is parallel to the $xy$ plane.

\[ x^2 + y^2 + z^2 - z^3 = 0 \]

\[ \nabla f = (2x, 2y, 2z - 3z^2) \]

need this parallel to $(0,0,1)$

so $x = 0$, $y = 0$

plug these values into the surface equation to get $z^2 = z^3$

or $z = 0, 1$

$\nabla f (0,0,0) = (0,0,0)$ which is not parallel to $(0,0,1)$

$\nabla f (0,0,1) = (0,0,-1)$ which is parallel to $(0,0,1)$ so

$P = (0,0,1)$
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