RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2019

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
- At the top of each book, clearly list the numbers of those problems appearing in the book and that you want to have graded. List them in order that they appear in the book.

Part I. Answer all questions.

1. Classify all finite groups G of order 2019 up to isomorphism. (Hints: The prime factors of 2019 are 3 and 673. Also, $255^3 - 1$ is divisible by 673.)

2. Let R be a ring (associative with 1) with finitely many elements. Prove that if R cannot be written as a direct product $R = R_1 \times R_2$ of smaller rings, then the number of elements of R is a power of a prime.

3. Let A be an $n \times n$ matrix over some field and let $f(t) = \det(A - tI_n)$ be its characteristic polynomial. Consider left multiplication by A

$$M \mapsto AM$$

as a linear transformation L_A on the space of $n \times n$ matrices. Prove that the characteristic polynomial of L_A is equal to $f(t)^n$.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let S be the subring of $\mathbb{C}[x, y]$ which consists of the polynomials $f(x, y)$ with $f(x, 0) = f(0, 0)$. Prove that S is not Noetherian.

5. Prove that for any prime p and any positive integer n, the group $GL(n, \mathbb{Z}/p\mathbb{Z})$ contains an element of order $(p^n - 1)$.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. Let $\mathcal{D}$ denote the open unit disc in $\mathbb{C}$, and let $\mathcal{U}$ denote the complement of the set of nonnegative real numbers in $\mathbb{C}$. Determine the set of all conformal maps $\mathcal{C}$ from $\mathcal{D}$ onto $\mathcal{U}$ which send 0 to -1.

2. Let $G = GL(2, \mathbb{R})$, the group of all invertible real 2×2 matrices $g = \left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$ (recall that membership in G is equivalent to $x_1x_4 - x_2x_3 \neq 0$). Endow G with the subspace topology of $\mathbb{R}^4$, via the embedding of the 2×2 matrix g as $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Let f be any continuous function of compact support on G and let $dg = (x_1x_4 - x_2x_3)^{-2}dx_1\, dx_2\, dx_3\, dx_4$. Prove that

$$\int_G f(hg)\, dg = \int_G f(g)\, dg$$

for any $h \in G$.

3. Let $\Re z > 1$ and $r \in \mathbb{R}$, and consider the integral

$$I_z(r) := \int_{\mathbb{R}} \frac{e^{-irx}}{(x^2+1)^z}\, dx.$$

a) Use the residue theorem to compute $I_2(r)$ for $r > 0$.

b) Show that for any fixed z and $0 < t < 1$, there exists a real number $C(z,t) > 0$ such that

$$|I_z(r)| \leq C(z,t)e^{-tr} \text{ for } r > 0.$$

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $\mathcal{U} = \{z \in \mathbb{C} | -\frac{3}{2} < \Re(z) < \frac{3}{2}\}$, and let $f : \mathcal{U} \to \mathbb{C}$ be a holomorphic function on $\mathcal{U}$ satisfying the identity

$$f(z+1) - 2f(z) + f(z-1) = 0$$

whenever the left-hand side is defined (that is, each of the three points $z-1$, z, and $z+1$ lie in $\mathcal{U}$).

a) Prove f extends to an entire function.

b) Use a variant of Liouville's theorem to prove that f cannot be bounded on $\mathcal{U}$ unless it is constant. (HINT: look at the sequence $f(z+n)$, where n is an integer.)

5. Let $\mathbb{H}$ denote the complex upper half plane, and let $f : \mathbb{H} \to \mathbb{H}$ denote a holomorphic function which has the property that $f\left(\frac{az+b}{cz+d}\right) = f(z)$ for some 2×2 rotation matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $GL(2, \mathbb{R})$. (That is, $g^t = g^{-1}$ and $\det(g) = 1$.) Suppose further that g^n is not the identity matrix for any integer n. Prove that f is constant. (HINT: study the behavior of f on the points $\{g^n z | n \in \mathbb{Z}\}$ where $g\, z = \frac{az+b}{cz+d}$. Alternatively, consider the Taylor series of $f(z)$ at the point $z = i$.)

End of Session 2

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2019

Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
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Part I. Answer all questions.

1. If $f : [0,1] \to (0,\infty)$ is absolutely continuous, must $1/f$ be? Prove it if so, and exhibit a counterexample if not.

2. Let (X, Σ, μ) be a measure space and let $f \in L^1(X,\mu)$. Show that for every $\epsilon > 0$, there exists $E \in \Sigma$ such that $\mu(E) < \infty$ and

$$\int_{X \setminus E} |f| d\mu < \epsilon.$$

3. Consider the space $X = \mathbb{R}$ endowed with distance function $d(x,y) = |e^x - e^y|$.

(a) Prove that d is a metric.

(b) Prove that (X,d) is not complete.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Evaluate the integral

$$\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dx dy,$$

making sure to justify every step.

5. Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

(a) Let $\{u_n\}_{n \geq 1}$ be a sequence in H so that $\sum_n \|u_n\| < \infty$. Prove that $\sum_n u_n$ converges in H.

(b) Suppose that $\|u_n\| \to \|u\|$ and $u_n \rightharpoonup u$ weakly, that is, $(u_n, v) \to (u,v)$, for all $v \in H$. Prove that $u_n \to u$ strongly, that is, $\|u_n - u\| \to 0$.

End of Session 3