

Homework 3, due Tuesday, February 12

I. Reading:

- Read do Carmo, Section 2-2.
- Read do Carmo, Appendix to Chapter 2, pages 125-133.
- Read the following comments.

Comments on Section 2.2 and Appendix for Chapter 2

The main goals for these two sections are

- (1) Understand the definition of a regular surface, its relation with and difference from the way a regular parametrized curve is defined.
- (2) Learn how to verify whether a given map from a domain in \mathbb{R}^2 to \mathbb{R}^3 is a parametrization for a surface.
- (3) Learn the concept of the differential of a map and its geometric interpretation.
- (4) Learn to use the Inverse Function Theorem.
- (5) Learn the concept of regular/singular value of a function and how to verify whether a given level surface of a function gives rise to a regular surface.

The following are some comments on these topics.

- When we defined a regular parametrized curve, we defined it to be a *map* α from an interval I to \mathbb{R}^n , $n = 2$ or 3 , such that (a) α is smooth, and (b) $\alpha'(t) \neq 0$ for all $t \in I$. Condition (a) allows us to use differential calculus, and condition (b) rules out sharp turning of tangent lines such as corners and cusps. Our definition of regular parametrized curves do allow self-intersections or “almost intersections” of the *trace* of such curves. Self-intersections of curves do happen frequently in describing physical phenomenon and including such curves in our study didn’t cause too much extra difficulty.

However, when dealing with surfaces, allowing self-intersections would cause considerable difficulty in some problems, so our definition of a *regular surface* would have two differences from our definition of a regular parametrized curve: (i) we will treat a surface as a subset of \mathbb{R}^n , $n = 2$ or 3 , instead of a map into \mathbb{R}^n ; (ii) we won’t allow the possibility of self-intersections—*however, we will also have a notion of parametrized surface and regular parametrized surface in 2.3. These correspond to the notion of a parametrized curve and regular parametrized curve, respectively.* But we still want to keep the differentiability and do not allow the sharp turning of “tangent planes”. Definition 1 on p.52 spells conditions that describe our requirements. It is a long and difficult definition. We will first give an initial analysis of this definition, and adopt a definition that is easier to work with, before we come back to give a more thorough discussion of the definition on p.52.

Note that even though a regular surface is defined as a subset of \mathbb{R}^n , its definition still involves maps from domains in \mathbb{R}^2 to \mathbb{R}^3 . The difference is that *it is not in terms of a single map, but a collection of maps whose appearance may depend on the location of the point p on the surface S .* Note next that condition 1 in Definition 1 corresponds to our condition (a) above for regular parametrized curves, and condition

3 will be seen to correspond to our condition (b) above. Condition 2 is put in to rule out self-intersections.

- We will first adopt a definition for a regular surface to correspond to Proposition 3 on p.63. Before we do that, let's first examine how a similar definition would work out for curves. Suppose $\alpha = \alpha(t) = (f(t), g(t), h(t)), t \in I$ is a regular parametrized curve. Then for any $t_0 \in I$, $\alpha'(t_0) \neq 0$ implies that one of $f'(t_0), g'(t_0), h'(t_0)$ is not zero. Suppose $f'(t_0) \neq 0$. Then by the Inverse Function Theorem in one-variable calculus, we can solve t in terms of x from $x = f(t)$ for t near t_0 and x near $x_0 := f(t_0)$: $t = f^{-1}(x)$. Substituting t in terms of $f^{-1}(x)$, the trace of α near t_0 , say, over $t \in I_{t_0} = (t_0 - \delta, t_0 + \delta)$, can also be described as

$$\alpha(I_{t_0}) = \{(x, g(f^{-1}(x)), h(f^{-1}(x))) \text{ for } x \text{ near } x_0\},$$

namely, as a graph over x : $y = g(f^{-1}(x))$ and $z = h(f^{-1}(x))$. To rule out possible self-intersection near $p_0 := \alpha(t_0)$, we require that for some $\epsilon > 0$, the part of $\alpha(I)$ inside the ball $B_\epsilon(p_0)$ with radius ϵ and center p_0 is just $\alpha(\tilde{I}_{t_0})$, but no more, where \tilde{I}_{t_0} is a slight modification of I_{t_0} . Written more symbolically,

$$B_\epsilon(p_0) \cap \alpha(I) = B_\epsilon(p_0) \cap \alpha(\tilde{I}_{t_0}) = \text{graph of } (y, z) \text{ over } x \text{ for } x \text{ in a neighborhood of } x_0.$$

- (*First Definition of a Regular Surface*). A subset S of \mathbb{R}^3 is called a regular surface, if for every point p on S , there is a neighborhood V of p in \mathbb{R}^3 such that $V \cap S$ is the graph of a differentiable (i.e. smooth) function over a domain in \mathbb{R}^2 . The function can take the form of $z = f(x, y)$, or $y = g(x, z)$, or $x = h(y, z)$.
- A surface is often given in terms of parametric representations, for instance, as $\alpha = \alpha(u, v) \in \mathbb{R}^3$ for (u, v) in some open domain U in \mathbb{R}^2 . To check whether this given parametric representation gives rise to a regular surface according to the above definition, we have to examine: (1)(local condition): whether near each parameter point $(u_0, v_0) \in U$, we can solve (u, v) in terms of one of the pairs of variables (x, y) , or (y, z) , (x, z) by a differentiable map, which would allow us to write the remaining variable in terms of this pair by substituting (u, v) in terms of them; and (2) (global condition) within a small neighborhood in \mathbb{R}^3 of $p_0 := \alpha(u_0, v_0)$ the only image of the parametric representation is this piece of graph in (1). Often, it is difficult, or impossible to carry out (1) explicitly. However, we may give a theoretical argument if the Inverse Function Theorem in multi-variable calculus can be applied.
- The Inverse Function Theorem involves the notion of the *differential* of a map. We first offer a definition from an analytical point of view and will later reconcile it with the more geometric definition on p. 127.

Let $F : U \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ and $x_0 \in U$. In the traditional multi-variable calculus context, F is said to be *differentiable* at x_0 , if there is a linear map from \mathbb{R}^n to \mathbb{R}^m , represented by an $m \times n$ matrix, A_{x_0} , with respect to the standard bases of \mathbb{R}^n and \mathbb{R}^m , such that

$$F(x_0 + h) - F(x_0) - A_{x_0}h = o(|h|), \text{ as } h \rightarrow 0 \text{ in } \mathbb{R}^n,$$

in the sense that

$$\lim_{h \rightarrow 0} \frac{F(x_0 + h) - F(x_0) - A_{x_0}h}{|h|} = 0, \text{ in } \mathbb{R}^m.$$

This A_{x_0} is called the differential (also Jacobian matrix) of F at x_0 , and is often given the notation $dF(x_0)$ as well. Recall that if $F = (f_1(x), \dots, f_m(x))$ is differentiable at x_0 , then A_{x_0} is uniquely determined in terms of the partial derivatives of $f_j(x)$ at x_0 : the (i, j) entry of A_{x_0} is

$$\frac{\partial f_i}{\partial x_j}(x_0), \quad i = 1, \dots, m, \quad j = 1, \dots, n.$$

- (*A Fact from Algebra*): A 3×2 matrix M whose two column vectors are $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ gives rise to a one-to-one linear map from \mathbb{R}^2 to \mathbb{R}^3 iff one of the following three equivalent conditions holds: (c1) $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, (c2) $\mathbf{u} \wedge \mathbf{v} \neq \mathbf{0}$, (c3) one of the determinants

$$\det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}, \quad \det \begin{pmatrix} u_1 & v_1 \\ u_3 & v_3 \end{pmatrix}, \quad \text{or} \quad \det \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}$$

is not zero.

- So to check whether we can apply the Inverse Function Theorem to a given parametric representation $\alpha = \alpha(u, v)$ near (u_0, v_0) , we need to check whether one of the the determinants at (u_0, v_0)

$$\frac{\partial(x(u, v), y(u, v))}{\partial(u, v)}, \quad \frac{\partial(x(u, v), z(u, v))}{\partial(u, v)}, \quad \frac{\partial(y(u, v), z(u, v))}{\partial(u, v)}$$

is not zero, which is equivalent to

$$\alpha_u(u_0, v_0) \wedge \alpha_v(u_0, v_0) \neq \mathbf{0},$$

by the algebraic fact above. This brings in condition 3 on p.52, as explained on p. 54.

- One reason we use the definition as given on p. 52, instead of sticking with the one based on Proposition 3 on p. 63, is that with the latter, one may have to pick many different local graphs to parametrize the surface, while the former may allow the possibility of fewer parametrizations, as shown by Example 1 on p. 55.
- When a set S is defined as the *level set* of a differentiable function $f(x, y, z)$: $S = \{(x, y, z) : f(x, y, z) = a\}$, and a is a regular value of f , then we know S is a regular surface by Proposition 2—this bypasses the potentially lengthy verifications required by either definition. The proof of Proposition 2 again uses the Inverse Function Theorem.
- The definition of *differential* on p. 127 is in terms of how the tangents to a curve at the domain are mapped to tangents to the image curve in the target. This will be very helpful in our geometric context. Study this definition carefully and have a good grasp on it. Here I will add some interpretations and rules that facilitate computations.

Let $n = 2$, $m = 1$. Verify that for the special cases $F_1(u, v) = u$ and $F_2(u, v) = v$, $du := dF_1(u, v)$ and $dv := dF_2(u, v)$ are defined, respectively, as linear maps $\mathbb{R}^2 \mapsto \mathbb{R}$ by

$$\begin{aligned} du : (1, 0) &\mapsto 1, & du : (0, 1) &\mapsto 0. \\ dv : (1, 0) &\mapsto 0, & dv : (0, 1) &\mapsto 1. \end{aligned}$$

We will endow du and dv such meanings. Now for a general differential map $F : U \subset \mathbb{R}^n \mapsto \mathbb{R}^m$, verify that

$$dF(u, v) = F_u(u, v)du + F_v(u, v)dv \text{ at every } (u, v) \text{ where } dF \text{ is defined,}$$

here, $F_u(u, v)$ and $F_v(u, v)$ denote the column vectors in \mathbb{R}^m whose j^{th} component are the partials of the j^{th} component of F with respect to u and v respectively. So we can give $dF(u, v)$ both a geometric meaning as given on p. 127 and an algebraic meaning as above—note the slight abuse of notation: $dF(u, v)$ itself is sometimes also used to denote the Jacobian matrix.

- (1) Let $(r, \theta) \mapsto (x, y)$ be given by

$$x = r \cos \theta, y = r \sin \theta.$$

Verify that

$$d \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} dr + \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix} d\theta = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

- (2) Reinterpret the chain rule (Proposition 8 on p. 129) in the situation of $(r, \theta) \mapsto (x, y) \mapsto f(x, y)$ as substitutions of the above relations into $df(x, y) = f_x(x, y)dx + f_y(x, y)dy$.
- (3) Verify that for any direction \mathbf{n} at (x, y) ,

$$df(\mathbf{n}) = \frac{\partial f}{\partial \mathbf{n}}(x, y).$$

- (4) From the substitutions above, find f_r, f_θ in terms of f_x and f_y .
- (5) At any (r, θ) , set $p = (x, y)$ where x and y are from the substitution rule above. Interpret f_r as $df_p(\mathbf{n}_r)$, where $\mathbf{n}_r = (\cos \theta, \sin \theta)$ is the unit radial vector at p , and f_θ as $df_p(\mathbf{n}_\theta)$, where $\mathbf{n}_\theta = (-r \sin \theta, r \cos \theta)$ is an angular vector at p orthogonal to \mathbf{n}_r . Note that \mathbf{n}_θ is not a unit vector.

II. Practice problems:

- (1) do Carmo, Section 2-2 (p.65–67), Exercises 1, 2, 3, 4, 7(a,b), 10, 11(b), 13.

III. Required problems:

- (1) do Carmo, Section 2-2 (p. 65), Exercise 5.
- (2) do Carmo, Section 2-2 (p. 66), Exercise 7(c).
- (3) do Carmo, Section 2-2 (p. 66), Exercise 8—you can't just quote the algebraic fact I listed earlier; in other words, you need to provide a proof for the algebra.
- (4) do Carmo, Section 2-2 (p. 66), Exercise 11(a).
- (5) do Carmo, Section 2-2 (p. 67), Exercise 15.
- (6) do Carmo, Section 2-2 (p. 67), Exercise 16.