640:350: LINEAR ALGEBRA HIGHLIGHTS OF LECTURES FOR WEEK 8

Sections to be covered: Summary review of Chapter 4 (Determinants)

(1) One of the most useful relations we learned in Chapter 3 was that

$$rank(A) = \dim(Col(A)) = \dim(Row(A)) = rank(A^t).$$

In deriving this relation, we used the relations between matrix A and its row echelon form R:

$$Row(A) = Row(R), \quad \text{and} \quad K_A = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\} = K_R = \{\mathbf{x} : R\mathbf{x} = \mathbf{0}\};$$

in addition,

$$\dim(Col(A)) = \dim(Col(R)), \quad \text{even thought } Col(A) \text{ may not equal } Col(R).$$

Using this relation, we provide a simple proof to
21(of section **3.2**). Since it is assumed that the $m \times n$ matrix A has rank m, we conclude that $\dim(Col(A)) = m$. But $Col(A)$ is a subspace of $\mathbb{R}^m$, which has dimension m itself. Thus $Col(A) = \mathbb{R}^m$. In particular, for each of the standard vector

$$\mathbf{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$j = 1, \ldots, m$, there must exist coefficients $b_{1j}, \ldots, b_{nj}$ such that

$$b_{1j}Col_1(A) + \ldots + b_{nj}Col_n(A) = \mathbf{e}_j.$$

Written in matrix form, these relations say

$$AB = \begin{pmatrix} \mathbf{e}_1 & \ldots \mathbf{e}_m \end{pmatrix} = I_m,$$

where B is the $n \times m$ matrix whose jth column is formed using the coefficients $b_{1j}, \ldots, b_{nj}$.

(2) One of the most useful features of determinants is their relations to (signed) area or volumns. If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^2$, then the determinant of the 2×2 matrix with $\mathbf{u}$ and $\mathbf{v}$ as the two row vectors is equal to the signed area of the parallelogram formed with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides. Because of this connection, we expect

$$\det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = -\det \begin{pmatrix} \mathbf{v} \\ \mathbf{u} \end{pmatrix},$$

$$\det \begin{pmatrix} c\mathbf{u} \\ \mathbf{v} \end{pmatrix} = c\det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix},$$

$$\det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} + c\mathbf{u} \end{pmatrix} = \det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

A geometric explanation for the last relation is that the parallelogram with $\mathbf{u}$ and $\mathbf{v} + c\mathbf{u}$ as adjacent sides has the same base and height as the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

(3) For an $n \times n$ matrix A, we would like to define its determinant so that it would carry similar algebraic properties. One way to make such a definition is to use the definition of cofactors and define the determinant of A in terms of the cofactor expansion as given in the text. The remaining task is to verify that the determinant as defined indeed obey the algebraic properties in terms of elementary row operations.

The text first develops Theorem 4.3 and the Lemma on p. 213 using mathematical induction, and use them to deduce the other properties, Theorem 4.4, Theorem 4.5, Theorem 4.6 and its corollary.

(4) For elementary matrices E, one can compute their determinants easily by using the properties on p. 217, and check directly that

$$\det(EA) = \det E \cdot \det A.$$

Using this properties repeatly, we have, for any elementary matrices $E_1, \ldots, E_k$,

$$\det(E_k \cdots E_1 A) = \det(E_k) \cdots \det(E_1) \det(A) = \det(E_k \cdots E_1) \det(A).$$

Since every invertible matrix P can be written as $E_k \cdots E_1$ for some elementary matrices $E_1, \ldots, E_k$, we have verified that

$$\det(PA) = \det(P) \cdot \det(A)$$

for invertible P. For P not invertible, $\det(P) = 0$, so the above relation continues to hold if we can verify that $\det(PA) = 0$ as well. This follows from $rank(PA) \leq rank(P) < n$, the latter because of $\det(P) = 0$. A similar argument establishes that

$$\det(A) = \det(A^t).$$

(5) Using the property that $\det(PA) = \det(P) \cdot \det(A)$ and the roles of elementary matrices in performing row operations, we conclude that $\det(A) = 0$ iff it has rank $< n$, equivalently, A is not invertible.

(6) It also follows from the property $\det(PA) = \det(P) \cdot \det(A)$ that for any invertible matrix Q, $\det(QAQ^{-1}) = \det(Q) \cdot \det(A) \cdot \det(Q^{-1}) = \det(A)$. Thus for any linear operator $T : V \mapsto V$, different matrix representations of T under different bases have equal determinants, as $[T]_\beta = Q[T]_\gamma Q^{-1}$ for some invertible Q. So we can use $\det([T]_\beta)$ under any basis β of V to define the determinant of the linear operator T, $\det(T)$. For any solid region S in V, $Vol(T(S)) = |\det(T)| Vol(S)$.