640:350: LINEAR ALGEBRA HIGHLIGHTS OF LECTURES FOR WEEK 7

Sections to be covered: Summary review of Chapter 3 (matrix operations and linear systems)

(1) Let E denote the elementary matrix obtained by performing an elementary row operation on the identity matrix I_n, then for any $n \times p$ matrix A, EA is the matrix obtained from A by performing the same elementary row operation on A; let F denote the elementary matrix obtained by performing an elementary column operation on the identity matrix I_p, then for any $n \times p$ matrix A, AF is the matrix obtained from A by performing the same elementary column operation on A

(2) Any elementary matrix is invertible, and its inverse is also an elementary matrix.

(3) In elementary linear algebra courses, the rank of a matrix A is often defined to be the number of pivots in the RREF of A. This definition requires two features: (i) the rank of a matrix A is not defined (computed) directly in terms of A, but one has to go through a reduction process to obtain a RREF; (ii) a matrix must have a unique RREF, or the number of pivots in the RREF should not depend on the potentially different RREFs, in order for this definition to make sense. In this course, the rank of a matrix is defined as the rank of the linear transformation L_A associated with the matrix A.

Recall that, for an $m \times n$ matrix A, the linear transformation $L_A : F^n \mapsto F^m$ is defined as $L_A(\mathbf{v}) = A\mathbf{v}$, for any $\mathbf{v} \in F^n$, and its rank is defined to be the dimension of the range of L_A. Theorem 2.2 says that rank(A) is equal to the dimension of $span\{L_A(\mathbf{v}_1), \cdots, L_A(\mathbf{v}_n)\}$, if $\{\mathbf{v}_1 \cdots, \mathbf{v}_n\}$ is a basis for F^n. If we take $\mathbf{v}_i = e_i$, $i = 1, \cdots, n$, to be the standard basis for F^n, then $L_A(\mathbf{v}_i)$ is the ith column of A. So the rank of A is the dimension of space spanned by the columns of A.

Here is how to reconcile these two definitions. It's fact that a matrix has a unique RREF. Let R denote the RREF of A. Then we have the following relation between A and R:

$$A\mathbf{x} = \mathbf{0} \quad \text{iff } R\mathbf{x} = \mathbf{0}.$$

Written in terms of the column vectors, this is

$$x_1 col_1(A) + \cdots + x_n col_n(A) = \mathbf{0} \quad \text{iff } x_1 col_1(R) + \cdots + x_n col_n(R) = \mathbf{0}.$$

The columns of R are classified into pivot columns and non-pivot columns, and the number of pivot columns is the elementary definition of the rank of A. The crucial link between A and R is that *those columns of A that correspond to the pivot columns of R are linearly independent and generate the span of the columns of A*, thus form a basis for the range of L_A. So the new definition of the rank of A agrees with the elementary definition.

(4) When we perform an elementary row operation on a matrix A to obtain a matrix B_1, then A and B_1 will have the same RREF, so they have the same rank according to the elementary definition of rank; yet the span of the columns of A may be different from the span of the columns of B_1, so it is not so direct to see why A and B_1 should have the same rank according to the new definition. However, our discussion above says that the span of the columns of A and the span of the columns of B_1 will have

the same dimension. Here is an example illustrating the situation.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_1 == \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then the column space of A is

$$span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

while the column space of B_1 is

$$span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Let B_2 be obtained from A by an elementary column operation, then A and B_2 will have the same column space, thus the same rank according to the new definition. However, A and B_2 may not have the same RREF, so it is not so direct to see that they would have the same rank according to the elementary definition. For example, we may have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

B_2 is obtained from A by adding the 3rd column of A to its 2nd column. A and B_2 are both RREF, so they have different RREF. Yet they have the same rank.