

Sections to be covered: 2.5

- (1) Suppose that $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\beta' = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ are two ordered bases for V , then any vector $\mathbf{v} \in V$ would have coordinate vectors $[\mathbf{v}]_\beta = [x_1, \dots, x_n]^T$ relative to β and $[\mathbf{v}]_{\beta'} = [y_1, \dots, y_n]^T$ relative to β' , respectively. By their definitions

$$\mathbf{v} = \sum_{i=1}^n x_i \mathbf{v}_i = [\mathbf{v}_1 \cdots \mathbf{v}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

and

$$\mathbf{v} = \sum_{i=1}^n y_i \mathbf{w}_i = [\mathbf{w}_1 \cdots \mathbf{w}_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Since β and β' are two bases, there exists Q_{ij} such that

$$\mathbf{w}_j = \sum_{i=1}^n Q_{ij} \mathbf{v}_i.$$

Written in matrix form, this is

$$[\mathbf{w}_1 \cdots \mathbf{w}_n] = [\mathbf{v}_1 \cdots \mathbf{v}_n]Q,$$

where $Q = (Q_{ij})$ is the change of coordinate matrix from β' to β . It now follows that

$$[\mathbf{v}_1 \cdots \mathbf{v}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [\mathbf{w}_1 \cdots \mathbf{w}_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [\mathbf{v}_1 \cdots \mathbf{v}_n]Q \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Q \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

- (2) When $T : V \mapsto V$ is a linear transformation, it has matrix representation $[T]_\beta$ and $[T]_{\beta'}$ relative to the bases β and β' , respectively. According to their definitions,

$$[T(\mathbf{v}_1) \cdots T(\mathbf{v}_n)] = [\mathbf{v}_1 \cdots \mathbf{v}_n][T]_\beta,$$

and

$$[T(\mathbf{w}_1) \cdots T(\mathbf{w}_n)] = [\mathbf{w}_1 \cdots \mathbf{w}_n][T]_{\beta'}.$$

But

$$[\mathbf{w}_1 \cdots \mathbf{w}_n] = [\mathbf{v}_1 \cdots \mathbf{v}_n]Q,$$

which leads to

$$[T(\mathbf{w}_1) \cdots T(\mathbf{w}_n)] = [T(\mathbf{v}_1) \cdots T(\mathbf{v}_n)]Q = [\mathbf{v}_1 \cdots \mathbf{v}_n][T]_\beta Q.$$

Comparing the two expressions for $[T(\mathbf{w}_1) \cdots T(\mathbf{w}_n)]$, we conclude that

$$[\mathbf{v}_1 \cdots \mathbf{v}_n][T]_\beta Q = [\mathbf{w}_1 \cdots \mathbf{w}_n][T]_{\beta'} = [\mathbf{v}_1 \cdots \mathbf{v}_n]Q[T]_{\beta'},$$

from which it follows that

$$[T]_{\beta}Q = Q[T]_{\beta'}.$$

We remark that Q , being a matrix representing the change between bases, is invertible. Therefore, we can write the relation above as

$$[T]_{\beta} = Q[T]_{\beta'}Q^{-1}.$$