640:350: LINEAR ALGEBRA HIGHLIGHTS OF LECTURES FOR WEEK 5

Sections to be covered: 2.2, 2.3, 2.4.

1. **Theorem 2.4** says that the property of a linear transformation being one-to-one is equivalent to its null space $N(T)$ being the trivial space $\{0\}$.
2. In the situation that T is a linear transformation from a vector space to another of *equal, finite* dimension, then $N(T) = \{0\}$ is equivalent to T being onto.
3. **Theorem 2.6** gives a way of defining a linear transformation, namely, through defining its action on a basis.
4. Discuss the notion of *ordered basis* and the *coordinate vector* of a vector $\mathbf{x}$ relative to an ordered basis $\beta = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$.
5. Suppose that $T : V \mapsto W$ is a linear transformation from V to W, and $\beta = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ and $\gamma = \{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$ are two ordered bases for V and W, respectively, then we can define a matrix $A = (a_{ij})$ by
$$T(\mathbf{v}_j) = \sum_{i=1}^{m} a_{ij}\mathbf{w}_i,$$
and find that, for any vector $\mathbf{x} \in V$,
$$[T(\mathbf{x})]_\gamma = A[\mathbf{x}]_\beta,$$
namely, the coordinate vector of $T(\mathbf{x})$ relative to γ is computed as A times the coordinate vector of $\mathbf{x}$ relative to β.
6. If $S, T : V \mapsto W$ are two linear transformations from V to W, then $S + T$ and aT are naturally defined, so the set $\mathcal{L}(V, S)$ of all linear transformations from V to W becomes a vector space under these operations. This is **Theorem 2.7**.
7. **Theorem 2.8** explains that relative to two ordered bases β and γ in V and W, respectively, the matrix representations of $S + T$, and aT are related to those of S and T as
   (a) $[S + T]_\beta^\gamma = [S]_\beta^\gamma + [T]_\beta^\gamma$ and
   (b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$.
8. If β and γ are two ordered bases of the same vector space V, and $\mathbf{x}$ is a vector in V, how does the coordinate vector $[\mathbf{x}]_\beta$ of $\mathbf{x}$ relative to β relate to the coordinate vector $[\mathbf{x}]_\gamma$ of $\mathbf{x}$ relative to γ? Suppose $T : V \mapsto V$ is a linear transformation from V to V, how does the matrix representation $[T]_\beta$ of T relative to β relate to the matrix representation $[T]_\gamma$ of T relative to γ? These questions will be explored in **2.5**. But here is an example. Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ denote the orthogonal projection of vectors in $\mathbb{R}^3$ onto the plane π through the origin with $\mathbf{n} = (n_1, n_2, n_3)$ as a unit normal vector. Then for any $\mathbf{x} \in \mathbb{R}^3$, $T(\mathbf{x}) = \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}$. So relative to the standard basis $\{\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)\}$,
$$\begin{aligned} T(\mathbf{e}_1) &= (1, 0, 0) - n_1(n_1, n_2, n_3) &&= (1 - n_2^2, -n_1n_2, -n_1n_3) \\ T(\mathbf{e}_2) &= (0, 1, 0) - n_2(n_1, n_2, n_3) &&= (-n_1n_2, 1 - n_2^2, -n_2n_3) \\ T(\mathbf{e}_3) &= (0, 0, 1) - n_3(n_1, n_2, n_3) &&= (-n_1n_3, -n_2n_3, 1 - n_3^3). \end{aligned}$$

Thus the matrix representation of T relative to the standard basis is

$$\begin{bmatrix} 1-n_2^2 & -n_1n_2 & -n_1n_3 \\ -n_1n_2 & 1-n_2^2 & -n_2n_3 \\ -n_1n_3 & -n_2n_3 & 1-n_3^3 \end{bmatrix}.$$

However, if we choose a pair of orthonormal vectors $\{\epsilon_1, \epsilon_2\}$ in the plane π, and use $\{\epsilon_1, \epsilon_2, \mathbf{n}\}$ as an order basis for $\mathbb{R}^3$, then

$$\begin{aligned} T(\epsilon_1) &= \epsilon_1 - (\epsilon_1 \cdot \mathbf{n})\mathbf{n} &&= \epsilon_1 \\ T(\epsilon_2) &= \epsilon_2 - (\epsilon_2 \cdot \mathbf{n})\mathbf{n} &&= \epsilon_2 \\ T(\mathbf{n}) &= (\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})\mathbf{n} &&= \mathbf{0}. \end{aligned}$$

Thus the matrix representation of T in this basis is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(9) In section **2.3** we define the composition of two linear transformations T and U, when $T : V \mapsto W$ and $U : W \mapsto Z$. The main theoreoms are **Theorems 2.11, 2.14**. A part of section **2.3** reviews the multiplication of two matrices and the properties of matrix multiplications. For example, **Theorems 2.12, 2.13, 2.16** are mainly reviewing properties of matrix multiplications.

(10) A more concrete way of reading the conclusion of **Theorem 2.14** is as follows. Let $\beta = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be an ordered basis for V, $\gamma = \{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ be an ordered basis for W, and $T : V \mapsto W$ be a linear transformation. Then the matrix $[T]_\beta^\gamma = [t_{ij}]$ is defined through

$$T(\mathbf{u}_j) = \sum_{i=1}^{m} t_{ij}\mathbf{v}_i, \quad \text{for } j = 1, \cdots, n,$$

or written in matrix notation

$$[T(\mathbf{u}_1) \cdots T(\mathbf{u}_n)] = [\mathbf{v}_1 \cdots \mathbf{v}_m][t_{ij}].$$

For any vector $\mathbf{u} \in V$, we can write $\mathbf{u}$ in the basis β

$$\mathbf{u} = \sum_{j=1}^{n} x_j\mathbf{u}_j = [\mathbf{u}_1 \cdots \mathbf{u}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [\mathbf{u}_1 \cdots \mathbf{u}_n][\mathbf{u}]_\beta.$$

Also there must exist $y_1, \cdots, y_m$ such that

$$T(\mathbf{u}) = \sum_{i=1}^{m} y_i\mathbf{v}_i = [\mathbf{v}_1 \cdots \mathbf{v}_m] \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

Then

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = [t_{ij}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [T]_\beta^\gamma \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

(11) **Section 2.4** deals with invertible linear transformations and reviews invertible matrices and link these to invertible linear transformations. Some basic facts are
- A linear transformation $T : V \mapsto W$ is invertible iff $N(T) = \{\mathbf{0}\}$ and $R(T) = W$.
- If $\dim V = \dim W$ are finite, then $T : V \mapsto W$ is invertible iff $N(T) = \{\mathbf{0}\}$, or $R(T) = W$, or simply $\dim R(T) = \dim W$.
- (Lemma on p.101) If $T : V \mapsto W$ is an invertible linear transformation from finite dimensional vector spaces V and W, then $\dim V = \dim W$.
- (Theorem 2.18) Let $T : V \mapsto W$ be a linear transformation, $[T]_\beta^\gamma$ be the matrix representation of T in some ordered bases β and γ of V and W, respectively, then T is invertible iff $[T]_\beta^\gamma$ is an invertible matrix. Furthermore, when T is invertible, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

(12) The remainder of **Section 2.4** discusses the concept of *isomorphism* between two vector spaces V and W. The focus here is the isomorphic relation between the two vector spaces, not the specific invertible linear transformation that makes them isomorphic. In particular we have
- (Theorem 2.19) Two finite dimensional vector spaces V and W are isomorphic to each other iff $\dim(V) = \dim(W)$.
- (Corollary) A vector space V over a field F is isomorphic to F^n iff $\dim(V) = n$.
- (Theorem 2.20) Let V and W be finite dimensional vector spaces of dimension n and m respectively, then the vector space $\mathcal{L}(V, W)$ of linear transformations from V to W is isomorphic to $M_{m\times n}(F)$.
- (Corollary) Let V and W be finite dimensional vector spaces of dimension n and m respectively, then the vector space $\mathcal{L}(V, W)$ has dimension mn.

(13) Here is a remark on how the dimension of a vector space depends on the specific field that is used. Consider $V = \{(c_1, \cdots, c_n) : c_i \in \mathbb{C}\}$. Then one can consider V as a vector space over the field $\mathbb{R}$ of real numbers, denoted as $(V, \mathbb{R})$, or as a vector space over the field $\mathbb{C}$ of complex numbers, denoted as $(V, \mathbb{C})$. Then the set β of standard vectors $\{e_1 = (1, \cdots, 0), \cdots, e_n = (0, \cdots, 1)\}$ in V can be proved to be linearly independent in both $(V, \mathbb{R})$ and $(V, \mathbb{C})$. However, β generates V over $\mathbb{C}$, thus forms a basis for $(V, \mathbb{C})$, but does not generate $(V, \mathbb{R})$. This can be seen as follows. Any vector $(c_1, \cdots, c_n) \in V$ can be written as

$$(c_1, \cdots, c_n) = c_1 e_1 + \cdots + c_n e_n, \quad \text{where } c_i \in \mathbb{C},$$

yet, the vector $(i, 0, \cdots, 0) \in V$ cannot be expressed as a linear combination of $e_1, \cdots, e_n$ with real coefficients. In fact, the set

$$\gamma = \{e_1 = (1, \cdots, 0), \cdots, e_n = (0, \cdots, 1), \eta_1 = (i, \cdots, 0), \cdots, \eta_n = (0, \cdots, i)\}$$

forms a basis for $(V, \mathbb{R})$. Thus $\dim_{\mathbb{R}}(V) = 2n$, while $\dim_{\mathbb{C}}(V) = n$.