640:350: LINEAR ALGEBRA HIGHLIGHTS OF LECTURES FOR WEEK 4

Sections to be covered: 2.1, 2.2, 2.3.

(1) Note that if A is an $m \times n$ matrix, then $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ defined by
$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for any vector } \mathbf{x} \in \mathbb{R}^n$$
is a linear transformation, yet for any non-zero vector $\mathbf{b}$,
$$\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$$
does not define a linear transformation.

(2) Note that both $T_1(f) = \int_a^b f(x)\,dx$ and $T_2(f) = \int_a^x f(x')\,dx'$ define linear transformations, but T_1 can be regarded as a linear transformation from $C[a,b]$ to $\mathbb{R}$, but T_2 has a different codomain: it can be regarded as a linear transformation from $C[a,b]$ to $C[a,b]$. The precise range of T_2 is in fact a subspace of $C[a,b]$: $T_2(C[a,b]) = \{g \in C^1[a,b] : g(a) = 0\}$.

(3) **Theorem 2.2** provides a method for finding a spanning set for the range $R(T)$ of a linear transformation T. Then the dimension and a basis of $R(T)$ can be found from the method of **Theorem 1.9**.

(4) The dimension of $R(T)$ is related to the dimension of the domain and the dimension of the null space $N(T)$ through the dimension theorem, **Theorem 2.3**.

(5) When a linear transformation T is defined through an $m \times n$ matrix A as
$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for any vector } \mathbf{x} \in \mathbb{R}^n$$
its $nullity(T)$ is the same of the nullity of the matrix A, which is equal to the # of free variables in solving $A\mathbf{x} = \mathbf{0}$, and its $rank(T)$ is the same as the rank of the matrix A, which is equal to the # of basic variables in solving $A\mathbf{x} = \mathbf{0}$. The dimension theorem is reflected as # of free variables + # of basic variables = n, the # of total variables, which is equal to the dimension of the domain. The general dimension theorem, **Theorem 2.3**, is proved directly, not through a matrix.